ELLIPSOMETRY

Stokes' parameters & related constructs
in optics & classical/quantum mechanics

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Introduction. Take a sting of length ℓ and pin its respective ends to the points $(+f, 0)$ and $(-f, 0)$ on the (x, y)-plane. Necessarily $2f \leqslant \ell$. Familiarly, the figure which results from the obvious "taut string condition" is an ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{1}$$

where

$$\begin{aligned} a &= \tfrac{1}{2}\ell \quad \text{is the semi-major axis} \\ b = \sqrt{a^2 - f^2} &= \tfrac{1}{2}\sqrt{\ell^2 - 4f^2} \leqslant a \quad \text{is the semi-minor axis} \end{aligned}$$

It is, in view of my present objectives, interesting to recall[1] that Maxwell's first publication—at fourteen, in the *Proceedings of the Royal Society of Edinburgh* —concerned an elaboration of this charming construction (which in practice does not work very well; it proves difficult to avoid parallax, the string stretches, pulls out the pins, saws the tip off the pencil).

Now pin one end of the string to the ceiling, and the other to a bob. You have constructed a pendulum with two degrees of freedom—an isotropic 2-dimensional oscillator—and observe that the bob traces what appears to be an ellipse, but an ellipse which precesses (and is, when you think about it, inscribed not on a plane but on a sphere of radius ℓ). The figure has been rendered this time not by a draftsman, but by God; i.e., by the laws of motion.

Look closely to an illuminated point marked on the plucked E-string of your double bass and you will observe that it traces a (wandering) ellipse. Circular

[1] C. W. F. Everitt, in *James Clerk Maxwell: Physicist & Natural Philosopher* (1975), tells the story (p. 47).

disks, when viewed obliquely from a distance, appear (in leading approximation) elliptical, and cast elliptical shadows. Ellipses surround us, in Nature (though you won't see many when you walk in the woods) and especially in the world of our own contrivance. Circles are the exception,[2] ellipses—failed circles—the rule.

In the celestial realm, the realm most distantly removed from our own contrivance, circles, in their perfection, were for a long time held to prevail. The reasons were mainly philosophical and æsthetic, though sun and moon do undeniably present themselves to us as circular disks. So fixated were natural philosophers on "the circle paradigm" that they were willing to embrace many-times-nested circles within circles—epicycles—in their effort to account for the astronomical data.[3] Kepler's 1st Law—his claim that the orbits traced by planets are in fact not epicyclic but elliptic—was, since it ran counter to such an entrenched tradition, radically revolutionary; it contained within it a whole new "natural æsthetic," and by implication shook to its foundations the natural philosopher's sense of how "well-designed worlds" might be constructed. In one relatively technical sense Kepler's idea strikes me still as so radical that in less familiar contexts I might, I confess, be inclined to dismiss such an idea as "implausible." For Kepler placed the sun at one focus *but nothing at the other*. His proposal embodies a broken symmetry, creates a preferred point with nothing to do.

Many of the ellipses encountered in scientific work today are what I might call "mental ellipses," artifacts of analytical discourse. For example: before me stands a chair. I have learned to associate with the chair a preferred point (its center of mass), and a symmetric matrix (the moment of inertia matrix). Associated with the latter is a crowd of ellipsoids/ellipses, all of which figure in my understanding of the chair and (were I to throw it out the window) its motion, but none of which is sensibly present in the chair, evident to the

[2] It is, in this light, curious that spheres—whether fashioned by surface tension, abrasion, central forces or some other agency—are, on the other hand, ubiquitous.

[3] We note in passing that ellipses are, from an epicyclic point of view, highly unnatural. The obvious model gives

$$\frac{(p\cos\theta+q\cos\phi)^2}{(p+q)^2}+\frac{(p\sin\theta+q\sin\phi)^2}{(p-q)^2}=1$$

and leads promptly to an equation of the form

$$\sum_{k=0}^{4}A_k(\theta;p,q)\cos^k\phi=0$$

The resulting $\phi(\theta;p,q)$ is almost too awkward to write out, and certainly not "pretty enough to be physically plausible."

non-analytical eye. To remark that in other contexts it becomes sometimes more difficult to distinguish

- ellipses evident to the senses from
- ellipses evident only to the prepared mind

is to remark a particular instance of a general circumstance: the distinction between "real attributes" and "imputed attributes" is—if defensible at all—frequently not entirely sharp. And it would, in the present instance, be to belabor a "distinction without a difference," for all ellipses, whether real or only imagined, are mathematically identical. It is in that surprisingly rich mathematics that a remarkable array of physical disciplines acquire a common interest, and an opportunity for fertile crosstalk. It is that crosstalk which lies at the center of my interest here, and which will motivate my account of the mathematics itself.

Whether the ellipses latent in a lightbeam are more—or less—sensible than those latent in a chair I am in no position to say. But this I can say: position yourself at inspection point P and, with the aid of a detector of exquisite resolution, look directly into an on-coming coherent/monochromatic beam. Maxwellian electrodyamics asserts that you will see[4] the mutually orthogonal $\mathbf{E}$ and $\mathbf{B}$ vectors to stand normal to the propagation vector $\mathbf{k}$, and to exhibit this time-dependence:

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}_1 \cos(\omega t + \delta_1) \\ \mathcal{E}_2 \cos(\omega t + \delta_2) \end{pmatrix} \tag{2}$$

With instruments of finite temporal resolution you will, however, detect actually not $\mathbf{E}(t)$ but only the figure which the flying $\mathbf{E}$-vector traces/retraces on the (E_1, E_2)-plane. That figure—got by eliminating t between $E_1(t)$ and $E_2(t)$—can be described

$$\mathcal{E}_2^2 E_1^2 - 2\mathcal{E}_1\mathcal{E}_2 \cos\delta \cdot E_1 E_2 + \mathcal{E}_1^2 E_2^2 = \mathcal{E}_1^2\mathcal{E}_2^2 \sin^2\delta \tag{3}$$

$$\delta \equiv \delta_2 - \delta_1 \equiv \text{ phase difference}$$

or again

$$\frac{1}{\mathcal{E}_1^2\mathcal{E}_2^2 \sin^2\delta} \cdot \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathcal{E}_2\mathcal{E}_2 & -\mathcal{E}_1\mathcal{E}_2\cos\delta \\ -\mathcal{E}_1\mathcal{E}_2\cos\delta & \mathcal{E}_1\mathcal{E}_1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 1 \tag{4}$$

And this—since

$$\det \begin{pmatrix} \mathcal{E}_2\mathcal{E}_2 & -\mathcal{E}_1\mathcal{E}_2\cos\delta \\ -\mathcal{E}_1\mathcal{E}_2\cos\delta & \mathcal{E}_1\mathcal{E}_1 \end{pmatrix} = \mathcal{E}_1^2\mathcal{E}_2^2(1-\cos^2\delta) = \mathcal{E}_1^2\mathcal{E}_2^2\sin^2\delta \geqslant 0 \tag{5}$$

—describes an ellipse: the light is said in the general case to be "elliptically polarized."

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[4] No! It asserts that you can use the following language to *account* for what you literally see.

Enter George Gabriel Stokes [1819–1903]. Stokes, who read mathematics while an undergraduate at Cambridge (where he spent his entire career, and where he became Lucasian Professor of Mathematics at age 30) and is perhaps best remembered for his mathematical accomplishments (Stokes' theorem, Stokes' phenomenon), but in life drew his mathematical inspiration mainly from his hands-on experimental activity. His early work was in hydrodynamics, from which he radiated into acoustics, whence into optics. He made contributions also to gravimetry and geophysics, meteorology, solar physics, chemistry and botany, and for many years to the administrative management of the Royal Society. His optical work, which in the 1840's related to the physical properties of the imagined "æther" and to diffractive phenomena, had by about 1851 come to focus on ellipsometry. The *Stokes parameters* which are our present concern were devised by Stokes as an aid to the interpretation of his experimental results, and were described[5] while Maxwell was still an undergraduate—well in advance of his formulation (1865) of the electromagnetic theory of light. It is, by the way, a curious fact that Stokes, who worked in so many areas, intentionally avoided electromagnetism (on the reported grounds that he considered that field to be well looked after by his good friend, William Thompson); his own approach to the parameters which bear his name must therefore have been markedly different from that adopted here; it was, I gather, frankly phenomenological, and by intention hewed close to the observational face of the physics.[6]

So Stokes' parameters (S_0, S_1, S_2, S_3) were born of optics, from the experimental study of polarizational phenomena. But they—together with certain attendant notions—relate (very usefully, as I hope to demonstrate) to ellipses generally, including especially those mined from deep below the surface of the physics. My ultimate objective here will be to explore their relation to the ellipses that arise from certain dynamical systems (especially oscillatory systems and the Kepler problem), and thus to reduce the element of mystery which still clings to some associated conservation laws. But I will allow myself to explore occasional side trails as we hike toward those intended camping spots.

My title is intended to underscore the potentially *broad applicability* of an idea originally introduced by Stokes to solve a rather narrowly conceived set of problems specific to optics. Optics will concern us, but does not lie at the focal point of what I have to say. In 1944 Alexandre Rothen coined the word "ellipsometer" to describe a device of his own invention, a modified "polarimeter" adapted to the study of thin films,[7] but that fact is so little known

[5] "On the composition and resolution of streams of polarized light from different sources," Trans. Camb. Phil. Soc. **9**, 399 (1852).

[6] For further details concerning Stokes' life and work, see C. C. Gillispie (editor), *Dictionary of Scientific Biography* (1976), Volume 13, pp. 74–79. The introductory "Historical Understanding of Polarized Light" in C. Brosseau, *Fundamentals of Polarized Light: A Statistic Optics Approach* serves well to place Stokes' accomplishment in broad perspective.

[7] E. Passaglia *et al* (editors), *Ellipsometry in the Measurement of Surfaces and Thin Films: Symposium Proceedings 1963*, NBS Misc. Pub. 256 (1964).

that I trust no confusion will result from my intention to use "ellipsometry" to mean "the mathematics of ellipses—broadly conceived."

1. Some elementary analytical geometry. Let real 2×2 matrices $\mathbb{U}$ and $\mathbb{R}$ be defined as follows:

$$\mathbb{U} = \begin{pmatrix} u & w \\ w & v \end{pmatrix} \qquad \text{and} \qquad \mathbb{R} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

$\mathbb{U}$ is a *symmetric* matrix, $\mathbb{R}$ a *proper rotation* matrix, and each is typical of its breed—generic. From

$$\begin{aligned} \det(\mathbb{U} - \lambda\mathbb{I}) &= \lambda^2 - (u+v)\lambda + (uv - w^2) \\ &= \lambda^2 - \operatorname{tr}\mathbb{U}\cdot\lambda + \det\,\mathbb{U} \end{aligned} \tag{6}$$

we learn that the eigenvalues of U can be described by the manifestly real expressions

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \left(\tfrac{u+v}{2}\right) \pm \sqrt{\left(\tfrac{u-v}{2}\right)^2 + w^2} \tag{7}$$

By computation

$$\mathbb{R}^{\mathsf{T}}\mathbb{U}\,\mathbb{R} = \begin{pmatrix} U & W \\ W & V \end{pmatrix} \tag{8}$$

with

$$\begin{aligned} U &= u\cos^2\psi + v\sin^2\psi + w2\cos\psi\sin\psi \\ &= \left(\tfrac{u+v}{2}\right) + \left[\left(\tfrac{u-v}{2}\right)\cos 2\psi + w\,\sin 2\psi\right] \end{aligned}$$

$$\begin{aligned} V &= u\sin^2\psi + v\cos^2\psi - w2\cos\psi\sin\psi \\ &= \left(\tfrac{u+v}{2}\right) - \left[\left(\tfrac{u-v}{2}\right)\cos 2\psi + w\,\sin 2\psi\right] \end{aligned}$$

$$\begin{aligned} W &= (v-u)\cos\psi\sin\psi + w(\cos^2\psi - \sin^2\psi) \\ &= -\left(\tfrac{u-v}{2}\right)\sin 2\psi + w\,\cos 2\psi \end{aligned}$$

from which it becomes evident that to *diagonalize* $\mathbb{R}^{\mathsf{T}}\mathbb{U}\,\mathbb{R}$ we have only to set

$$\tan 2\psi = \frac{w}{\frac{1}{2}(u-v)} \tag{9}$$

For ψ thus determined, the terms which survive on the diagonal of (8) are precisely the *eigenvalues* of $\mathbb{U}$, as described by (7):

$$\mathbb{R}^{\mathsf{T}}\mathbb{U}\,\mathbb{R} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{10}$$

We conclude that the curve

$$\mathbf{x}^{\mathsf{T}}\mathbb{U}\mathbf{x} = ux_1^2 + 2wx_1x_2 + vx_2^2 = 1$$

can, by rotation of the coordinate system (write $\mathbf{x} = \mathbb{R}\mathbf{y}$), be brought to the canonical form

$$\lambda_1 \cdot y_1^2 + \lambda_2 \cdot y_2^2 = 1$$

which describes

an **ellipse** if λ_1 and λ_2 are both positive
an **hyperbola** if λ_1 and λ_2 are of opposite signs

and becomes degenerate in all other cases. We restrict our attention henceforth to the elliptic case; the results obtained above then admit of the geometrical interpretation shown in Figure 1.

The so-called "eccentricity"—defined

$$\begin{aligned} e &\equiv \tfrac{\text{distance between foci}}{\text{length of major axis}} \\ &= \frac{f}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - (b/a)^2} = \sqrt{1 - (\lambda_2/\lambda_1)} \end{aligned}$$

—provides a standard measure of the "shape" of an ellipse; one has

$$\text{eccentricity} = \begin{cases} 0 \text{ for fat ellipses (circles)} \\ 1 \text{ for flat ellipses (line segments)} \end{cases}$$

Another (and for our purposes more useful) measure of shape emerges when one looks to *the population of rectangles which can be circumscribed about a given ellipse*. In the circular case $e = 0$ these are all squares, of area $A = (2a)^2$ and (semi)diagonal measure $d = \sqrt{a^2 + a^2}$. In the opposite limit ($e = 1$) the circumscribing rectangles assume all possible proportions; they range in area from $A = 0$ to $A = \frac{1}{2}a^2$ (the latter pertains to the circumscribing square), but their (semi)diagonal measure—since the ellipse in all cases *constitutes* the diagonal—is in all cases the same: $d = a = \sqrt{a^2 + 0^2}$. One can show in the general case (I won't, but will let F. L. Griffin[8] and Figure 2 tell the story) that

- The rectangles which can be circumscribed about an ellipse all have the same (semi)diagonal measure $d = \sqrt{a^2 + b^2}$; their vertices (to say the same thing another way) all lie on a circle of radius d.
- The rectangle of least area is the "principal rectangle"—the rectangle aligned with the principal axes of the ellipse. It has area $A = 4ab$, and a "shape" of which $\chi = \arctan(b/a)$ provides a convenient measure.

[8] See §264 in F. L. Griffin's *Mathematical Analysis: Higher Course* (1927).

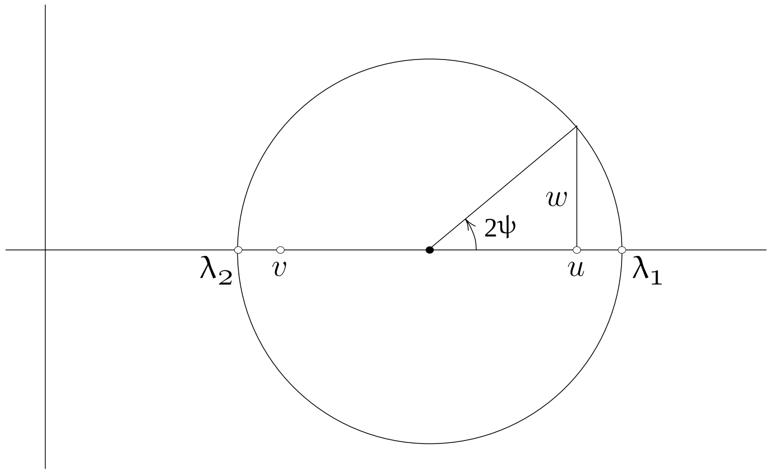

FIGURE 1: *Graphical construction of the angle ψ through which the coordinate system must be rotated in order to achieve*

$$\begin{pmatrix} u & w \\ w & v \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The figure derives from (9), and forms the basis of what is known to engineers as "Mohr's construction." As u grows smaller λ_1 becomes negative: the elliptic case has become hyperbolic.

Since

$$d = \sqrt{a^2 + b^2} \qquad \text{serves to characterize the } \textbf{size} \tag{12.1}$$

$$\psi = \tfrac{1}{2}\arctan\left\{\tfrac{2w}{u-v}\right\} \qquad \text{serves to characterize the } \textbf{orientation} \tag{12.2}$$

$$\chi = \arctan\left\{\tfrac{b}{a}\right\} \qquad \text{serves to characterize the } \textbf{shape} \tag{12.3}$$

of the principal rectangle, they serve to characterize also the ellipse which it circumscribes. To ellipses which are kinematically/dynamically generated we must also assign a **chirality**, indicative of the clockwise/counterclockwise sense with which we are to consider the ellipse to have been traced in time.

The attentive reader will have noticed that (12.2) is formulated in terms of the *matrix elements* of $\mathbb{U}$, while (12.1) and (12.3) refer in effect to the *spectral* properties of $\mathbb{U}$. I turn now to the removal of this formal defect. Looking first to (12.1), we have

$$d^2 = a^2 + b^2 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{\lambda_1 + \lambda_2}{\lambda_1 \cdot \lambda_2}$$

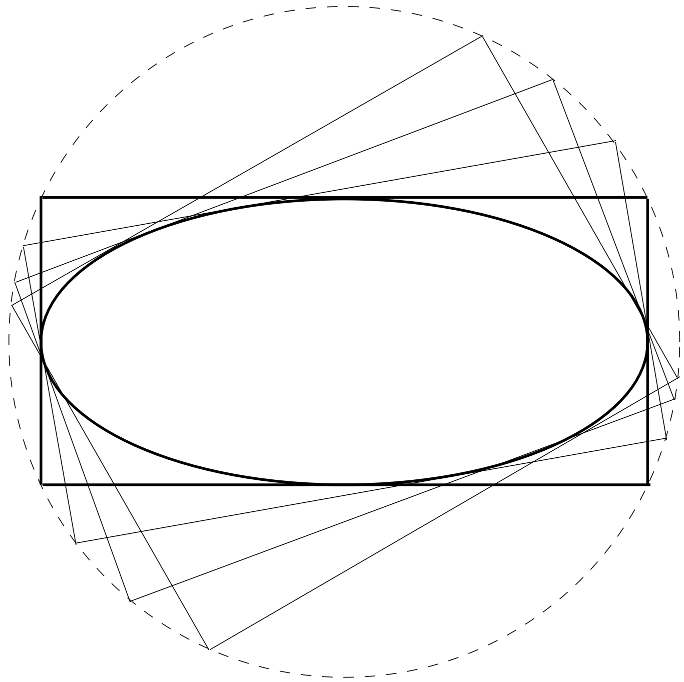

FIGURE 2: *Rectangles circumscribed about an ellipse all have the same diagonal measure* $d = \sqrt{a^2 + b^2}$. *The rectangle of least area is aligned with the principal axes of the ellipse. The shape of the enveloped ellipse—most commonly described by the "ellipticity"—can usefully be associated with the shape of the principal rectangle (slope of its diagonal).*

and drawing upon (7) obtain

$$d^2 = \frac{u+v}{uv - w^2} = \frac{\mathrm{tr}\,\mathbb{U}}{\det \mathbb{U}} \tag{13}$$

Looking next to (12.3), we have $b/a = \sqrt{\lambda_2/\lambda_1} = \tan\chi$, and it becomes therefore natural to write

$$\lambda_2/\lambda_1 = \tan^2\chi$$

Now a bit of a trick: we recall that

$$\sin 2\chi = \frac{2\tan\chi}{1+\tan^2\chi}$$

and use the preceeding result to obtain

$$\sin 2\chi = 2\frac{\sqrt{\lambda_1\cdot\lambda_2}}{\lambda_1+\lambda_2} = 2\frac{\sqrt{\det\mathbb{U}}}{\mathrm{tr}\,\mathbb{U}} = 2\frac{\sqrt{uv-w^2}}{u+v}$$

So we have

$$d = \sqrt{\frac{u+v}{uv-w^2}} \tag{14.1}$$

$$\tan 2\psi = 2\,\frac{w}{u-v} \tag{14.2}$$

$$\sin 2\chi = 2\,\frac{\sqrt{uv-w^2}}{u+v} \tag{14.3}$$

which serve to describe size, orientation and shape (d, ψ and χ) directly in terms of the matrix elements of $\mathbb{U}$. We are not surprised to notice that size (d) and shape (χ) can be described in terms of rotationally-invariant properties of $\mathbb{U}$ (i.e., in terms of $\operatorname{tr}\mathbb{U}$ and $\det\mathbb{U}$) while orientation (ψ) cannot be, but are a bit surprised to notice that

$$\begin{aligned}\big[\tfrac{1}{2}(u-v)\tan 2\psi\big]^2 + \big[\tfrac{1}{2}(u+v)\sin 2\chi\big]^2 &= w^2 + (uv - w^2)\\ &= uv\\ &= \big[\tfrac{1}{2}(u+v)\big]^2 - \big[\tfrac{1}{2}(u-v)\big]^2\end{aligned}$$

This curious result can be written

$$(u+v)^2 = (u-v)^2 + (u-v)^2\tan^2 2\psi + (u+v)^2\sin^2 2\chi$$

or more simply

$$\mathcal{S}_0^2 = \mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2 \tag{15}$$

provided we use

$$\left.\begin{aligned}\mathcal{S}_0 &\equiv \pm(u+v)\\ \mathcal{S}_1 &\equiv \pm(u-v)\\ \mathcal{S}_2 &\equiv \pm(S_1\tan 2\psi) = \pm 2w\\ \mathcal{S}_3 &\equiv \pm(S_0\,\sin 2\chi) = \pm 2\sqrt{uv-w^2}\end{aligned}\right\} \tag{16}$$

to introduce $\mathcal{S}$-parameters which remain for the moment *determined only to within independent sign specifications*. From

$$\mathcal{S}_1^2 + \mathcal{S}_2^2 = \begin{cases}\mathcal{S}_1^2(1+\tan^2 2\psi) = \mathcal{S}_1^2/(\cos 2\psi)^2\\[2ex] \mathcal{S}_0^2 - \mathcal{S}_3^2 = \mathcal{S}_0^2(1-\sin^2 2\chi) = \mathcal{S}_0^2\cos^2 2\chi\end{cases}$$

(which hold for *all* sign specifications) we obtain

$$\mathcal{S}_1 = \pm\mathcal{S}_0\cos 2\chi\cos 2\psi$$

whence

$$\mathcal{S}_2 = \pm(\mathcal{S}_0\cos 2\chi\cos 2\psi)\tan 2\psi = \pm\mathcal{S}_0\cos 2\chi\,\sin 2\psi$$

Equations (16) can therefore be formulated

$$\left.\begin{aligned} \mathcal{S}_0 &= \pm(u+v) \\ \mathcal{S}_1 &= \pm(u-v) && = \pm\mathcal{S}_0 \cos 2\chi \cos 2\psi \\ \mathcal{S}_2 &= \pm 2w && = \pm\mathcal{S}_0 \cos 2\chi \sin 2\psi \\ \mathcal{S}_3 &= \pm 2\sqrt{uv-w^2} && = \pm\mathcal{S}_0 \sin 2\chi \end{aligned}\right\} \tag{17}$$

In (17) we have succeeded in associating ellipses of assorted sizes, orientations, shapes (and, as will emerge, chiralities) with the points of—by (15)— a certain cone in a nameless place which we might call "4-dimensional $\mathcal{S}$-space".

In view of (17) it becomes natural to introduce dimensionless[9] variables

$$\left.\begin{aligned} {}_1 &\equiv \mathcal{S}_1/\mathcal{S}_0 = \pm\cos 2\chi \cos 2\psi \\ {}_2 &\equiv \mathcal{S}_2/\mathcal{S}_0 = \pm\cos 2\chi \sin 2\psi \\ {}_3 &\equiv \mathcal{S}_3/\mathcal{S}_0 = \pm \sin 2\chi \end{aligned}\right\} \tag{18}$$

which serve to associate ellipses of assorted orientations, shapes *but irrespective of scale* with points on a unit sphere in Euclidean 3-space. This representation —historically the point of departure for an elegantly powerful train of formal elaborations due to Poincaré[10]—proves advantageous in situations (and there are many) in which either

- scale is an irrelevant feature of the ellipses in which we have interest, or
- scale has been set once and for all by some circumstance special to the application at hand.

The "Poincaré sphere," in relation to the ellipses it was designed to represent, is shown in Figure 3, and results from (18) when—which is our option—all signs are made positive. Note especially that the parameters ψ and χ which describe elliptic orientation and shape have *acquired factors of 2* in their spherical coordinate interpretations.

We have come at this point to the threshhold—but only the threshhold— of some wonderful applied mathematics, much of it due to Poincaré, some of more recent invention. I propose, however, to let the applications motivate the further development of the theory, and turn now therefore to the optics which gave birth to this subject.

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[9] The parameters $\mathcal{S}$ are, it will be noted, dimensionally exotic:

$$\begin{aligned} [\,\mathcal{S}\,] &= \text{dimensionality of } u,\ v \text{ and } w \\ &= \tfrac{1}{\text{area}} \text{ for ellipses drawn on paper} \end{aligned}$$

[10] *Théorie Mathématique de la Lumière* (1892), Vol. 2, p. 275. We examine some of the details in §7.

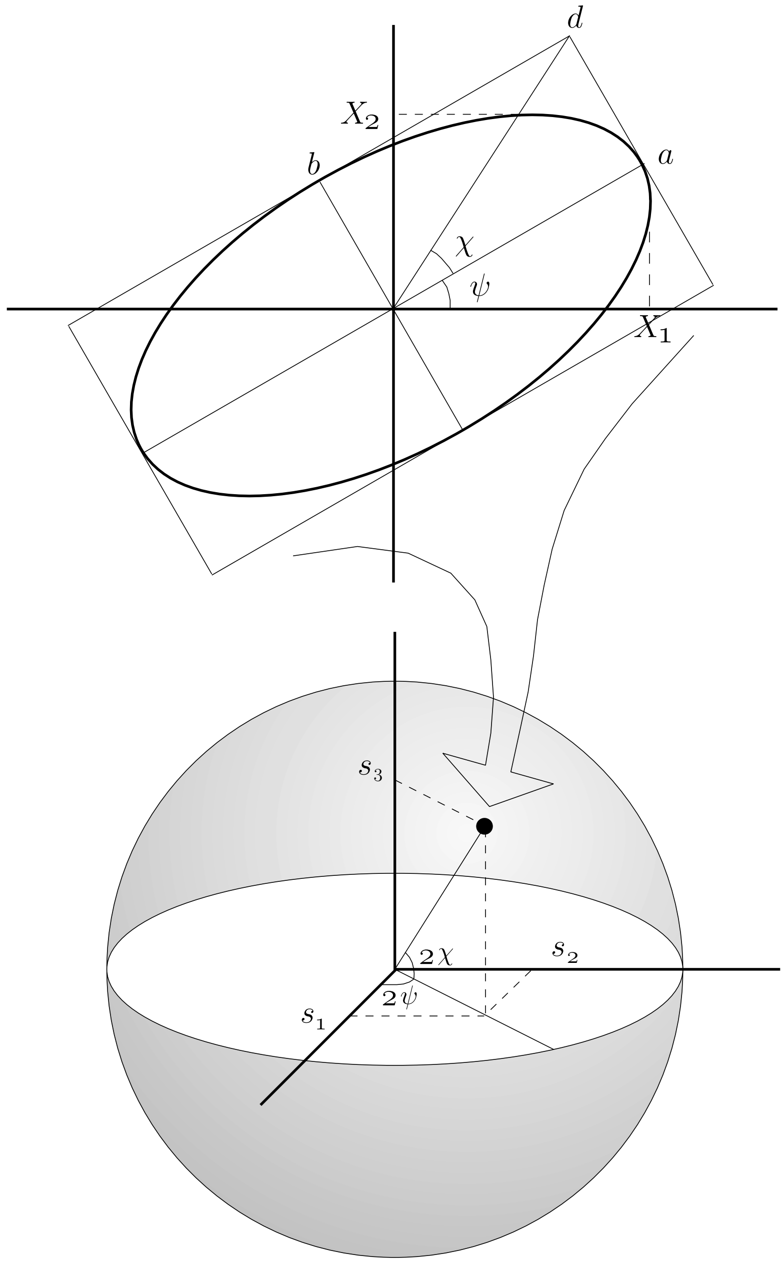

FIGURE 3: *Poincaré sphere: (18) has been used to associate points in the (ψ, χ)-parameterized space of all oriented ellipses (irrespective of scale) with points on the unit sphere in 3-dimensional -space. By "oriented ellipse" I understand an ellipse to which a circulation sense has been assigned, a "handedness"—geometrical analog of physical "chirality."*

PART I: APPLICATIONS TO OPTICS

2. Monochromatic beam description. At (4) we encountered ellipses drawn not "on paper" but on the electric E-plane—ellipses which can be described

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u & w \\ w & v \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 1$$

with

$$\left.\begin{aligned} u &= +D\cdot\mathcal{E}_2\mathcal{E}_2 \\ v &= +D\cdot\mathcal{E}_1\mathcal{E}_1 \\ w &= -D\cdot\mathcal{E}_1\mathcal{E}_2\cos\delta \end{aligned}\right\} \tag{19.1}$$

$$D \equiv 1/\mathcal{E}_1^2\mathcal{E}_2^2\sin^2\delta \tag{19.2}$$

Returning with this information to (17) we have

$$\begin{aligned} \mathcal{S}_0 &= \pm D\cdot(\mathcal{E}_2^2+\mathcal{E}_1^2) \\ \mathcal{S}_1 &= \pm D\cdot(\mathcal{E}_2^2-\mathcal{E}_1^2) \\ \mathcal{S}_2 &= \pm D\cdot 2\mathcal{E}_1\mathcal{E}_2\cos\delta \\ \mathcal{S}_3 &= \pm D\cdot 2\mathcal{E}_1\mathcal{E}_2\sin\delta \end{aligned}$$

"Stokes' parameters" are called into being by conventional resolution of the sign ambiguities and abandonment of the determinental factor D:

$$\left.\begin{aligned} S_0 &= \mathcal{E}_1^2+\mathcal{E}_2^2 \\ S_1 &= \mathcal{E}_1^2-\mathcal{E}_2^2 &&= S_0\cos 2\chi\cos 2\psi \\ S_2 &= 2\mathcal{E}_1\mathcal{E}_2\cos\delta &&= S_0\cos 2\chi\sin 2\psi \\ S_3 &= 2\mathcal{E}_1\mathcal{E}_2\sin\delta &&= S_0\sin 2\chi \end{aligned}\right\} \tag{20}$$

"Abandonment of the D-factors" preserves intact the fundamental relation (15), which now reads

$$S_0^2 = S_1^2+S_2^2+S_3^2 \tag{21}$$

and is motivated by a physical consideration about which Stokes himself could have had only an inkling: one does not directly "see" the orbit traced out by the tip of the $\mathbf{E}$-vector; rather, one measures (with a photometer of slow temporal resolution) the[11]

$$\begin{aligned} \text{optical beam intensity} &\equiv \begin{cases}\text{time-averaged rate per unit area with} \\ \text{which energy is delivered to the detector}\end{cases} \\ &= \langle\,\text{Poynting vector: } \mathbf{S}=c(\mathbf{E}\times\mathbf{B})\,\rangle \\ &\sim \mathcal{E}_1^2+\mathcal{E}_2^2 \quad \text{for monochromatic beams} \end{aligned}$$

[11] See ELECTRODYNAMICS (1972), p. 421.

It was precisely because the parameters (S_0, S_1, S_2, S_3) relate so directly to the *observational* realities of optical beams—to the analysis and description of beams and of the action of devices that transform beams—that they first recommended themselves to Stokes' (pre-Maxwellian) intuition.[12]

To gain some sense of how Stokes' construction works in the context for which it was originally intended, consider first the optical signal

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}_1 \cos(\omega t + \delta_1) \\ 0 \end{pmatrix}$$

Such a signal is "linearly polarized in the 1-direction." Since $\mathcal{E}_2 = 0$ (relative phase δ is therefore undefined) the Stokes quartet can, according to (20), be described

$$S \equiv \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1^2 \\ \mathcal{E}_1^2 \\ 0 \\ 0 \end{pmatrix}$$

while the Poincaré vector

$$\equiv \begin{pmatrix} S_1/S_0 \\ S_2/S_0 \\ S_3/S_0 \end{pmatrix} = \begin{pmatrix} \cos 2\chi \cos 2\psi \\ \cos 2\chi \, \sin 2\psi \\ \sin 2\chi \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{in this instance} \tag{23.1}$$

and can, in an obvious sense, be said "to puncture the equator of the Poincaré sphere at an axial point." Associated with that point is (see again (12)) an

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[12] It will become clear as we proceed that Stokes built better than he could have known. Stokes' construction, as transmogrified by Poincaré, has revealed itself to be not only one of the most economically designed and sharpest tools in the work-a-day optician's tookchest, but also to be remarkably robust; it arises in a natural way from *statistical* optics, and has therefore useful things to say about the properties of non-idealized "natural" lightbeams. Stokes' construction anticipates (by more than 70 years!) much that we now recognize to be most characteristic—both formally and philosophically—of the quantum theory. Nor is this, in retrospect, too surprising; both optics and quantum mechanics treat linear systems—systems dominated by a "principle of superposition"—the observable properties of which are described by quadratic constructions. The Stokes/Poincaré formalism relates more particularly to the quantum theory of angular momentum (spin), which is well known to be the intersection-point of a rich constellation of deep mathematical ideas. For an excellent brief account of the latter ramifications see §2–8 of J. M. Jauch & F. Rorhlich, *The Theory of Photons & Electrons* (1955), to which I owe my own first introduction to Stokes' parameters.

ellipse whose

$$\begin{aligned} \text{shape parameter } \chi &= 0 \\ \text{orientation parameter } \psi &= 0 \end{aligned} \tag{23.2}$$

which is another way of saying what we assumed at the outset: the beam is "linearly $\longleftrightarrow$ polarized." If the beam were, on the other hand, linearly $\updownarrow$ polarized we would have obtained

$$\mathbf{E}(t) = \begin{pmatrix} 0 \\ \mathcal{E}_2 \cos(\omega t + \delta_2) \end{pmatrix}$$

giving

$$S = \begin{pmatrix} \mathcal{E}_2^2 \\ -\mathcal{E}_2^2 \\ 0 \\ 0 \end{pmatrix} \quad \text{whence} \quad = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{cases} \chi = 0 \\ \psi = 90° \end{cases} \tag{24}$$

The -vectors encountered above puncture the Poincaré sphere at diametrically opposite points, and—it is important to notice—represent beams which are, in a familiar sense "oppositely polarized." Such beams, when superimposed, *do not interfere*. Linear polarization in the general case arises from setting $\delta = 0$; then

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}_1 \cos(\omega t + \delta_1) \\ \mathcal{E}_2 \cos(\omega t + \delta_1) \end{pmatrix} \tag{25.1}$$

gives

$$S = \begin{pmatrix} \mathcal{E}_1^2 + \mathcal{E}_2^2 \\ \mathcal{E}_1^2 - \mathcal{E}_2^2 \\ 2\mathcal{E}_1\mathcal{E}_2 \\ 0 \end{pmatrix} \tag{25.2}$$

which if we write

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{E} \cos\psi \\ \mathcal{E}_2 &= \mathcal{E} \sin\psi \end{aligned} \tag{25.3}$$

becomes

$$S = \mathcal{E}^2 \cdot \begin{pmatrix} 1 \\ \cos 2\psi \\ \sin 2\psi \\ 0 \end{pmatrix} \tag{25.4}$$

giving

$$= \begin{pmatrix} \cos 2\psi \\ \sin 2\psi \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{cases} \chi = 0 \\ \psi = \arctan(\mathcal{E}_2/\mathcal{E}_1) \end{cases} \tag{25.5}$$

from which the preceeding examples can be recovered as special cases. Evidently

$$\begin{array}{c} \text{equatorial points on} \\ \text{the Poincaré sphere} \end{array} \iff \begin{array}{l} \text{linearly polarized beams} \\ \text{of all orientations} \end{array} \tag{26}$$

In particular,

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{entails} \quad \begin{cases} \chi = 0 \\ \psi = 45° \end{cases}$$

To describe a circularly polarized beam we write

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}\cos(\omega t + \delta_1) \\ \mathcal{E}\cos(\omega t + \delta_1 + \delta) \end{pmatrix} \quad \text{and set} \quad \delta = 90° \tag{27.1}$$

Then

$$S = \begin{pmatrix} 2\mathcal{E}^2 \\ 0 \\ 0 \\ 2\mathcal{E}^2 \end{pmatrix} \quad \text{gives} \quad = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{whence} \quad \begin{cases} \chi = 45° \\ \psi = \text{undefined} \end{cases} \tag{27.2}$$

To reverse the sign of δ (i.e., to set $\delta = -90°$) is to reverse also the sign of the Poincaré vector (move from the north pole to the south pole of the Poincaré sphere): physically, to *reverse the chirality* of the wave. The point of general interest here is that

> *The Stokes/Poincaré representations distinguish between ellipses of identical figure but opposite orientation, and physically between waves which differ only with respect to their chirality.*

While the results reported above do serve to illustrate how simply and directly Stokes' parameters relate to what might be called "the state description problem" for idealized lightbeams, they convey little sense either of the ease with which the parameters characteristic of a beam can be measured in the laboratory or of the remarkable variety of the optical problems which their use serves almost automatically to illuminate—little sense of the "robustness" of Stokes' construction, or of its physical "naturalness." In following sections I will discuss, in sequence,

- ideas associated with the physical determination of the Stokes parameters characteristic of a beam;
- an elegant characterization of the action of beam-modification devices;
- the relation of Stokes' construction to the leading statistical properties of real lightbeams.

3. Monochromatic beam analysis. Discussion of the first of those topics proceeds from a couple of very simple physical ideas, but is notable for the mathematical complexity which tends at every turn to intrude (a complexity which the Stokes parameters serve in every instance to render transparent). Looking to the first of those "very simple physical ideas": polarizers accomplish their work by (i) resolving the incident beam into components characteristic of the device, and then (ii) differentially attenuating those components

$$\mathbf{E}_{\text{in}}(t) = \mathbf{E}_a(t) + \mathbf{E}_b(t) \xrightarrow[\text{typical}]{} \mathbf{E}_{\text{out}}(t) = k_a \cdot \mathbf{E}_a(t) + k_b \cdot \mathbf{E}_b(t)$$

The "perfect" polarizers to which we restrict our theoretical attention have the property that they are *transparent* to one compontent but *extinguish* the other

$$k_a = 1 \quad \text{and} \quad k_b = 0$$

The action of such idealized devices

$$\mathbf{E}_{\text{in}}(t) = \mathbf{E}_a(t) + \mathbf{E}_b(t) \xrightarrow[\text{ideal polarizer}]{} \mathbf{E}_{\text{out}}(t) = \mathbf{E}_a(t) \tag{28}$$

is, in an evident sense, "projective;" the output of such a device would pass unaffected through a second (identical) such device. The analytical problem central to the theory of such devices has to do—as (28) makes clear—with the identification/description of the relevant "pre-adapted basis." Suppose, for example, we had in mind a linear polarizer of arbitrary alignment ψ. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{f}_1, \mathbf{f}_2\}$—of which the former refers conventionally to what might be called the "alignment of the lab bench" and the latter to the alignment of the polarizer—stand in this relatively rotated relationship:

$$\begin{aligned}\mathbf{e}_1 &= \cos\psi\cdot\mathbf{f}_1 - \sin\psi\cdot\mathbf{f}_2\\ \mathbf{e}_2 &= \sin\psi\cdot\mathbf{f}_1 + \cos\psi\cdot\mathbf{f}_2\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}(t) &= [\mathcal{E}_1\cos(\omega t+\delta_1)]\cdot\mathbf{e}_1 + [\mathcal{E}_2\cos(\omega t+\delta_2)]\cdot\mathbf{e}_2\\ &= [\mathcal{E}_1\cos(\omega t+\delta_1)]\cdot[\cos\psi\cdot\mathbf{f}_1 - \sin\psi\cdot\mathbf{f}_2]\\ &\qquad + [\mathcal{E}_2\cos(\omega t+\delta_2)]\cdot[\sin\psi\cdot\mathbf{f}_1 + \cos\psi\cdot\mathbf{f}_2]\\ &= [\mathcal{F}_1\cos(\omega t+\theta_1)]\cdot\mathbf{f}_1 + [\mathcal{F}_2\cos(\omega t+\theta_2)]\cdot\mathbf{f}_2\end{aligned}$$

is found by elementary calculation to entail

$$\left.\begin{aligned}\mathcal{F}_1\cos\theta_1 &= \quad\mathcal{E}_1\cos\delta_1\cos\psi + \mathcal{E}_2\cos\delta_2\sin\psi\\ \mathcal{F}_1\sin\theta_1 &= \quad\mathcal{E}_1\sin\delta_1\cos\psi + \mathcal{E}_2\sin\delta_2\sin\psi\\ \mathcal{F}_2\cos\theta_2 &= -\mathcal{E}_1\cos\delta_1\sin\psi + \mathcal{E}_2\cos\delta_2\cos\psi\\ \mathcal{F}_2\sin\theta_2 &= -\mathcal{E}_1\sin\delta_1\,\sin\psi + \mathcal{E}_2\sin\delta_2\cos\psi\end{aligned}\right\} \tag{29}$$

Immediately

$$\begin{aligned}\mathcal{F}_1^2 &= \mathcal{E}_1^2\cos^2\psi + \mathcal{E}_1\mathcal{E}_2\cos(\delta_1-\delta_2)\sin 2\psi + \mathcal{E}_2^2\sin^2\psi\\ \mathcal{F}_2^2 &= \mathcal{E}_1^2\sin^2\psi - \mathcal{E}_1\mathcal{E}_2\cos(\delta_1-\delta_2)\sin 2\psi + \mathcal{E}_2^2\cos^2\psi\end{aligned} \tag{30}$$

so we have

$$\left.\begin{aligned}\mathcal{F}_1 &= \sqrt{\mathcal{E}_1^2\cos^2\psi + \mathcal{E}_1\mathcal{E}_2\cos(\delta_1-\delta_2)\sin 2\psi + \mathcal{E}_2^2\sin^2\psi}\\ \mathcal{F}_2 &= \sqrt{\mathcal{E}_1^2\sin^2\psi - \mathcal{E}_1\mathcal{E}_2\cos(\delta_1-\delta_2)\sin 2\psi + \mathcal{E}_2^2\cos^2\psi}\\ \theta_1 &= \arctan\left\{\frac{\mathcal{E}_1\sin\delta_1\cos\psi + \mathcal{E}_2\sin\delta_2\sin\psi}{\mathcal{E}_1\cos\delta_1\cos\psi + \mathcal{E}_2\cos\delta_2\sin\psi}\right\}\\ \theta_2 &= \arctan\left\{\frac{\mathcal{E}_1\sin\delta_1\,\sin\psi - \mathcal{E}_2\sin\delta_2\cos\psi}{\mathcal{E}_1\cos\delta_1\sin\psi - \mathcal{E}_2\cos\delta_2\cos\psi}\right\}\end{aligned}\right\} \tag{31}$$

These equations, though they relate directly to the description of the physical **E**-vector, are notable for their surprising complexity. Much simpler relations result when one looks to certain naturally-emergent *quadratic combinations* of physical variables; we notice that

$$\left.\begin{aligned}\mathcal{F}_1^2+\mathcal{F}_2^2 &= \mathcal{E}_1^2+\mathcal{E}_2^2\\ \mathcal{F}_1^2-\mathcal{F}_2^2 &= (\mathcal{E}_1^2-\mathcal{E}_2^2)\cos 2\psi+2\mathcal{E}_1\mathcal{E}_2\cos\delta\,\sin 2\psi\\ 2\mathcal{F}_1\mathcal{F}_2\cos\theta &= -(\mathcal{E}_1^2-\mathcal{E}_2^2)\,\sin 2\psi+2\mathcal{E}_1\mathcal{E}_2\cos\delta\cos 2\psi\\ 2\mathcal{F}_1\mathcal{F}_2\sin\theta &= 2\mathcal{E}_1\mathcal{E}_2\sin\delta\end{aligned}\right\}\tag{32}$$

It is striking that $\{\mathbf{e}_1,\mathbf{e}_2\}\longrightarrow\{\mathbf{f}_1,\mathbf{f}_2\}$ sets up a *linear* relationship among the quadratic constucts in question, and striking also that they are precisely the constructs to which Stokes has directed our attention; so natural—both physically and mathematically—has been their emergence that I suspect we have reproduced here Stokes' original train of thought. Equations (32) can be notated

$$\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&\cos 2\psi&\sin 2\psi&0\\0&-\sin 2\psi&\cos 2\psi&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}\tag{33}$$

where (see again (20))

$$\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}=\begin{pmatrix}\mathcal{E}_1^2+\mathcal{E}_2^2\\ \mathcal{E}_1^2-\mathcal{E}_2^2\\ 2\mathcal{E}_1\mathcal{E}_2\cos\delta\\ 2\mathcal{E}_1\mathcal{E}_2\sin\delta\end{pmatrix}\quad\text{are Stokes' parameters relative to the }\mathbf{e}\text{-basis}$$

while

$$\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}=\begin{pmatrix}\mathcal{F}_1^2+\mathcal{F}_2^2\\ \mathcal{F}_1^2-\mathcal{F}_2^2\\ 2\mathcal{F}_1\mathcal{F}_2\cos\theta\\ 2\mathcal{F}_1\mathcal{F}_2\sin\theta\end{pmatrix}\quad\text{are Stokes' parameters relative to the }\mathbf{f}\text{-basis}$$

The action of a linear polarizer whose alignment coincides with that of the $\mathbf{f}_1$-axis is—relative to the **f**-basis (this being, of course, the whole point of "basis pre-adaptation"!)—very easy to describe, and has, in effect, already been described; in essence, $\mathcal{F}_2\longrightarrow 0$, which gives

$$\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}_{\text{in}}\xrightarrow[\text{polarizer co-aligned with }\mathbf{f}_1\text{-axis}]{}\begin{pmatrix}\mathcal{F}_1^2\\ \mathcal{F}_1^2\\0\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(P_0+P_1)\\ \frac{1}{2}(P_0+P_1)\\0\\0\end{pmatrix}\equiv\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}_{\text{out}}$$

or again

$$\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}_{\text{out}}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0&0\\ \frac{1}{2}&\frac{1}{2}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}P_0\\P_1\\P_2\\P_3\end{pmatrix}_{\text{in}}$$

Using (33) to obtain from this a statement referent of the original **e**-basis we (after some elementary matrix multiplication) obtain

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \xrightarrow[\text{linear polarizer at } \psi^\circ]{} \tfrac{1}{2}[S_0 + S_1\cos 2\psi + S_2 \sin 2\psi] \begin{pmatrix} 1 \\ \cos 2\psi \\ \sin 2\psi \\ 0 \end{pmatrix} \tag{34}$$

Of which the following are notable special cases:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \xrightarrow[\psi=0^\circ]{} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac{1}{2}(S_0+S_1) \\ \frac{1}{2}(S_0+S_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \tag{35.1}$$

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \xrightarrow[\psi=45^\circ]{} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac{1}{2}(S_0+S_2) \\ 0 \\ \frac{1}{2}(S_0+S_2) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \tag{35.2}$$

Drawing finally upon (20) we observe, in connection with (34), that

$$\begin{aligned} &\tfrac{1}{2}[S_0 + S_1\cos 2\psi + S_2\sin 2\psi] \\ &\qquad = \tfrac{1}{2}S_0[1 + \cos 2\chi_0(\cos 2\psi_0 \cos 2\psi + \sin 2\psi_0 \sin 2\psi)] \\ &\qquad = \tfrac{1}{2}S_0[1 + \cos 2\chi_0 \cdot \cos 2(\psi - \psi_0)] \end{aligned}$$

where the subscripted angles refer to the state of the input beam. When the input beam is already linearly polarized one has $\chi_0 = 0$, and in the special case $\psi_0 = \psi$ one easily recovers (25.4); the linear polarizer has in fact accomplished what it designed to accomplish.

Looking now to the relatively unfamiliar case of *circular* polarization, we write

$$\left.\begin{aligned} \mathbf{E}_{\circlearrowright}(t) &= [\tfrac{1}{\sqrt{2}}\mathcal{A}\cos(\omega t + \alpha)]\,\mathbf{e}_1 - [\tfrac{1}{\sqrt{2}}\mathcal{A}\sin(\omega t + \alpha)]\,\mathbf{e}_2 \\ \mathbf{E}_{\circlearrowleft}(t) &= [\tfrac{1}{\sqrt{2}}\mathcal{B}\cos(\omega t + \beta)]\,\mathbf{e}_1 + [\tfrac{1}{\sqrt{2}}\mathcal{B}\sin(\omega t + \beta)]\,\mathbf{e}_2 \end{aligned}\right\} \tag{36}$$

(which describe circularly polarized beams of opposite chirality) and form

$$\begin{aligned} \mathbf{E}(t) &= \mathbf{E}_{\circlearrowright}(t) + \mathbf{E}_{\circlearrowleft}(t) \\ &= \tfrac{1}{\sqrt{2}}\big[(\mathcal{A}\cos\alpha + \mathcal{B}\cos\beta)\cos\omega t - (\mathcal{A}\sin\alpha + \mathcal{B}\sin\beta)\sin\omega t\big]\mathbf{e}_1 \\ &\quad - \tfrac{1}{\sqrt{2}}\big[(\mathcal{A}\cos\alpha - \mathcal{B}\cos\beta)\sin\omega t + (\mathcal{A}\sin\alpha - \mathcal{B}\sin\beta)\cos\omega t\big]\mathbf{e}_2 \end{aligned}$$

Comparison with the more standard representation

$$\begin{aligned} \mathbf{E}(t) &= [\mathcal{E}_1\cos(\omega t + \delta_1)]\mathbf{e}_1 + [\mathcal{E}_2\cos(\omega t + \delta_2)]\mathbf{e}_2 \\ &= [\mathcal{E}_1\cos\delta_1\cos\omega t - \mathcal{E}_1\sin\delta_1\sin\omega t]\mathbf{e}_1 \\ &\qquad + [\mathcal{E}_2\cos\delta_2\cos\omega t - \mathcal{E}_2\sin\delta_2\sin\omega t]\mathbf{e}_2 \end{aligned} \tag{37}$$

gives

$$\left.\begin{aligned}\mathcal{A}\cos\alpha + \mathcal{B}\cos\beta &= \sqrt{2}\mathcal{E}_1\cos\delta_1\\ \mathcal{A}\cos\alpha - \mathcal{B}\cos\beta &= \sqrt{2}\mathcal{E}_2\,\sin\delta_2\\ \mathcal{A}\sin\alpha + \mathcal{B}\sin\beta &= \sqrt{2}\mathcal{E}_1\,\sin\delta_1\\ \mathcal{A}\sin\alpha - \mathcal{B}\sin\beta &= \sqrt{2}\mathcal{E}_2\cos\delta_2\end{aligned}\right\}\tag{38}$$

and inversely

$$\left.\begin{aligned}\sqrt{2}\mathcal{A}\cos\alpha &= \mathcal{E}_1\cos\delta_1 - \mathcal{E}_2\,\sin\delta_2\\ \sqrt{2}\mathcal{A}\,\sin\alpha &= \mathcal{E}_1\,\sin\delta_1 + \mathcal{E}_2\cos\delta_2\\ \sqrt{2}\mathcal{B}\,\cos\beta &= \mathcal{E}_1\cos\delta_1 + \mathcal{E}_2\,\sin\delta_2\\ \sqrt{2}\mathcal{B}\,\sin\beta &= \mathcal{E}_1\,\sin\delta_1 - \mathcal{E}_2\cos\delta_2\end{aligned}\right\}\tag{39}$$

Working from (38) we obtain

$$\left.\begin{aligned}\sqrt{2}\mathcal{E}_1 &= \sqrt{\mathcal{A}^2 + 2\mathcal{A}\mathcal{B}\cos\gamma + \mathcal{B}^2}\\ \sqrt{2}\mathcal{E}_2 &= \sqrt{\mathcal{A}^2 - 2\mathcal{A}\mathcal{B}\cos\gamma + \mathcal{B}^2}\\ \tan\delta_1 &= +\frac{\mathcal{A}\sin\alpha + \mathcal{B}\sin\beta}{\mathcal{A}\cos\alpha + \mathcal{B}\cos\beta}\\ \tan\delta_2 &= -\frac{\mathcal{A}\cos\alpha - \mathcal{B}\cos\beta}{\mathcal{A}\sin\alpha - \mathcal{B}\sin\beta}\end{aligned}\right\}\tag{40.1}$$

with $\gamma \equiv \alpha - \beta$, while (39) by similar arguments gives

$$\left.\begin{aligned}\sqrt{2}\mathcal{A} &= \sqrt{\mathcal{E}_1^2 + 2\mathcal{E}_1\mathcal{E}_2\sin\delta + \mathcal{E}_2^2}\\ \sqrt{2}\mathcal{B} &= \sqrt{\mathcal{E}_1^2 - 2\mathcal{E}_1\mathcal{E}_2\sin\delta + \mathcal{E}_2^2}\\ \tan\alpha &= \frac{\mathcal{E}_1\,\sin\delta_1 - \mathcal{E}_2\cos\delta_2}{\mathcal{E}_1\cos\delta_1 + \mathcal{E}_2\,\sin\delta_2}\\ \tan\beta &= \frac{\mathcal{E}_1\,\sin\delta_1 + \mathcal{E}_2\cos\delta_2}{\mathcal{E}_1\cos\delta_1 - \mathcal{E}_2\,\sin\delta_2}\end{aligned}\right\}\tag{40.2}$$

Equations (40) permit interconversion between the "standard representation" and the "counter-rotational representation" of $\mathbf{E}(t)$; they are notable for their formal complexity (great simplifications will be achieved in §6), and for the fact that they make it utterly natural to write (as Stokes was the first to do)

$$\left.\begin{aligned}\mathcal{E}_1^2 + \mathcal{E}_2^2 &= S_0 = \mathcal{A}^2 + \mathcal{B}^2\\ \mathcal{E}_1^2 - \mathcal{E}_2^2 &= S_1 = 2\mathcal{A}\mathcal{B}\cos\gamma\\ 2\mathcal{E}_1\mathcal{E}_2\cos\delta &= S_2 = 2\mathcal{A}\mathcal{B}\,\sin\gamma\\ 2\mathcal{E}_1\mathcal{E}_2\,\sin\delta &= S_3 = \mathcal{A}^2 - \mathcal{B}^2\end{aligned}\right\}\tag{41}$$

The action of a circular polarizer which passes $\mathbf{E}_{\circlearrowright}$ but extinguishes $\mathbf{E}_{\circlearrowleft}$ is, in the latter representation, very easy describe (this again being the whole point of "basis pre-adaptation"); in essence, $\mathcal{B} \longrightarrow 0$, which gives (compare (35))

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \xrightarrow[\circlearrowleft\text{ polarizer}]{\hspace{5em}} \begin{pmatrix} \mathcal{A}^2 \\ 0 \\ 0 \\ \mathcal{A}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(S_0+S_3) \\ 0 \\ 0 \\ \frac{1}{2}(S_0+S_3) \end{pmatrix}$$

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \tag{42}$$

We are in position now to draw upon the second of the "very simple physical ideas" mentioned at the beginning of this discussion—an idea which has to do with the operation not of perfect polarizers but of perfect photometers. Such devices—which (because I find it convenient to absorb a trivial factor into their calebration) I shall call "J-meters"—measure the time-averaged energy flux or *intensity* of the incident lightbeam; in effect, they examine of the incident beam $\{S_0, S_1, S_2, S_3\}$ and announce the value of S_0. Consider now the following protocol: a monochromatic beam

- falls *unobstructed* upon a J-meter, which registers J_0;
- is obstructed by a 0°-*linear polarizer* while *en route* to the J-meter, which registers J_1;
- is obstructed by a 45°-*linear polarizer* while *en route* to the J-meter, which registers J_2;
- is obstructed by a $\circlearrowright$-*circular polarizer* while *en route* to the J-meter, which registers J_3;

Drawing upon (35) and (42) we have

$$\begin{aligned} J_0 &= S_0 \\ J_1 &= \tfrac{1}{2}(S_0+S_1) \\ J_2 &= \tfrac{1}{2}(S_0+S_2) \\ J_3 &= \tfrac{1}{2}(S_0+S_3) \end{aligned}$$

giving

$$\left.\begin{aligned} S_0 &= J_0 \\ S_1 &= 2J_1 - J_0 \\ S_2 &= 2J_2 - J_0 \\ S_3 &= 2J_3 - J_0 \end{aligned}\right\} \tag{43}$$

which establishes the *direct observability* of Stokes' parameters. We notice that for monochromatic beams—but not, as will emerge, for statistically more complex beams—one of the J-measurements can, in fact, be omitted, for

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \quad\Rightarrow\quad J_0^2 = 2\big\{J_1(J_0-J_1) + J_2(J_0-J_2) + J_3(J_0-J_3)\big\}$$

4. Beam manipulation. We have already noted the naturalness and utility of statements of the form

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \longrightarrow \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \tag{44}$$

When first encountered—at (35.1)—the matrix $\mathbb{M}$ was structured in such a way as to represent the action of a certain linear polarizer. Curiously, it was not Stokes himself but H. Muller who—in the 1940's (which is to say: nearly a century after Stokes laid the foundations of our subject, and half a century after Poincaré's elaboration of it)—first appreciated[13] how general is the utility of (49), and worked out (but never published, except to his students at MIT) the basic implications of this thought:

> Just as, and to the same extent that, the Stokes parameters $\{S_0, S_1, S_2, S_3\}$ serve to describe/characterize lightbeams, so also, and to that same extent, do the Muller matrices $\mathbb{M}$ serve to describe/characterize the beam-modification properties of optical elements.

The door is thus opened to the creation—as an exercise in applied linear algebra!—of a "general theory of optical elements," insofar as the action of the elements in question has to do with manipulation of the intensity/polarization characteristics of the transmitted beam.[14] It becomes natural to say of elements that they are "equivalent" if their Mueller matrices are identical. The analysis of sequenced elements becomes an exercise in matrix multiplication:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \mathbb{M}_n \cdots \mathbb{M}_2 \mathbb{M}_1 \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}}$$

And attention is directed quite naturally to the formulation of certain "realizability" conditions; for example, if $\mathbb{M}$ is to be realizable as a passive optical element then it must be the case that

$$\text{output intensity} \leqslant \text{input intensity}$$

for all inputs; i.e., that (for all possible assignments of value to $\{S_0, S_1, S_2, S_3\}$)

$$M_{00}S_0 + M_{01}S_1 + M_{02}S_2 + M_{03}S_3 \leqslant S_0$$

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[13] That's the legend, but not quite true; the same idea had been put forward by P. Soleillet in "Sur les paramètres caractérisant la polarisation partielle de la lumière dan les phénomènes de florescence," Ann. Physique **12**, 23 (1929), which nobody read.

[14] For the sketched outlines of the general theory to which I allude, see CLASSICAL ELECTRODYNAMICS (1980), pp. 353–363.

Similar in spirit is this observation: suppose it to be a property of the optical element we have in mind that

$$\text{monochromatic input} \Longrightarrow \text{monochromatic output}$$

For such elements it will be invariably the case that

$$(S|S)_{\text{out}} = (S|S)_{\text{in}} = 0 \tag{45}$$

where I have adopted the notation

$$(S|S) \equiv S_0^2 - S_1^2 - S_2^2 - S_3^2 = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}^{\mathsf{T}} \mathbb{G} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

where

$$\mathbb{G} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Condition (45) imposes upon the elements of $\mathbb{M}$ a set of conditions which we can in these notations express

$$\mathbb{M}^{\mathsf{T}}\mathbb{G}\,\mathbb{M} = \mu \cdot \mathbb{G} \tag{46}$$

where necessarily $\mu = \sqrt{\det \mathbb{M}}$. It is a matter of deep interest (or at least of curiosity) that (46) serves to define the so-called *conformal group*, of which (set $\mu = 1$) the 4-dimensional *Lorentz group* is an important subgroup. The latter is, of course, fundamental to special relativity, while the former is (for that reason) fundamental to the relativistic dynamics of massless particles (of which photons provide in present context the most natural example; we are, after all, concerned with the physics of light beams!) and of associated field theories (most notably electrodynamics). The implications of (46) have, for these reasons, been exhaustively studied; the wonderful fact is that all of the resulting large body of mathematical knowledge and technique stands now instantly available—accidentally, as it were; ready-made and free of charge—to the theoretical optician. Wonderful also is the fact that

> "Experiments in 4-dimensional relativity" (for example: demonstration of the emergence—actually the optical analog of the emergence—of "Thomas precessional effects" from the composition of non-collinear boosts[15]) can now be carried out

[15] The fact that, in the non-collinear case,

$$(\text{boost}) \cdot (\text{boost}) = (\text{boost}) \cdot (\text{rotational factor})$$

was missed by Einstein. T. Y. Thomas once told me that he himself learned of the fact from A. S. Eddington (see p. 99 of his *Theory of Relativity* (1924)), but it appears to have been first noticed either by W. de Sitter or J. A. Schouten.

on a 1-dimensional optical bench—with nothing in motion (except, of course, for the lightbeam!).

Though this is not the place for a systematic account of what has come to be called the "Muller calculus," I would, by way of illustration, like to return briefly to the Muller matrix

$$\mathbb{P} \equiv \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By quick calculation $\mathbb{P}^{\mathsf{T}}\mathbb{G}\,\mathbb{P} = \mathbb{O}$, which by $\det \mathbb{P} = 0$ is consistent with (46). Quick calculation establishes also that $\mathbb{P}$ is projective in the sense that $\mathbb{P}^2 = \mathbb{P}$. From $\det(\mathbb{P} - \lambda\mathbb{I}) = (\lambda - 1)\lambda^3$ we learn that $\mathbb{P}$ has eigenvalues $\{1, 0, 0, 0\}$, and conclude that $\mathbb{P}$ projects onto a 1-dimensional subspace of 4-dimensional Stokes space; actually—this being, in fact, an implication of (35.1)—

$$\mathbb{P} \quad \text{projects onto the ray} \quad S\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

In physical language, $\mathbb{P}$ is the Muller matrix representative of a "polarizer" of such design that it is transparent to (and only to) incident beams of type $\{S, S, 0, 0\}$. One should, of course, not draw from this single example the conclusion that all optical elements are by nature "polarizers." The Muller matrix

$$\mathbb{M} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}$$

—realizable as a passive device if and only if $0 \leqslant k \leqslant 1$—is consistent with (46) but non-projective (therefore not a polarizer), and describes the action of a "neutral filter." Non-projective Muller matrices of the type

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix}$$

($\mathbb{R}$ is a 3×3 rotation matrix: $\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I}$) have the property that they in all cases *preserve the intensity* of the transmitted beam; their action is to accomplish a *rotation of the Poincaré sphere*. Such "rotators" are transparent to beams of the description

$$\begin{pmatrix} S_0 \\ \mathbf{S} \end{pmatrix} \quad \text{where} \quad \begin{cases} \mathbf{S} \text{ is the real eigenvector of } \mathbb{R}\text{: } \mathbb{R}\,\mathbf{S} = \mathbf{S} \\ S_0 = \sqrt{\mathbf{S}\cdot\mathbf{S}} \text{ provides radian measure of the rotation} \\ \hat{\mathbf{S}} \text{ identifies the "rotational axis" in Stokes space} \end{cases}$$

and their physical realizations are called "retarders."[16]

[16] See E. Hecht, *Optics* (2nd edition, 1987), §8.7.

5. Quasi-monochromatic beams. We have proceeded thus far on the basis of the unrealistic presumption that "monochromatic lightbeams"—beams in which the **E**-vector traces and with perfect regularity retraces ellipses of perfect precision—provide reasonable approximations to the lightbeams encountered in optical laboratories. And it was the elementary physics of such idealized beams that led to the introduction of Stokes' parameters, and to their illustrative application. But in point of physical fact the beams encountered in laboratories are complex entities, describable as *superpositions* of monochromatic beams, but more usefully thought of as trains of superimposed pulses. On a time scale short in comparison to its "coherence time" the physical beam will present the aspect of a monochromatic beam, but as pulses are replaced by successor pulses the variables $\mathcal{E}_1, \mathcal{E}_2, \delta_1, \delta_2$ will acquire a time-dependence—slow relative to the coherence time, but fast relative to the response time of a photometer. The ellipse traced by the $\mathbf{E}(t)$-vector will be seen (or would be, if only our eye were quick enough) to undergo slow deformation, to degenerate into a fuzzy figure which may remain "somewhat elliptical on the average" but may retain no detectable structure at all. Such a beam, since it contains necessarily Fourier components of *various* frequencies, cannot properly be called "monochromatic." But if the bandwidth is sufficiently narrow then one can expect to be able to write (compare (2))

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}_1(t)\cos(\omega t + \delta_1(t)) \\ \mathcal{E}_2(t)\cos(\omega t + \delta_2(t)) \end{pmatrix} \tag{47}$$

to describe the "slow meander of the path traced by the rapidly flying spot."[17] Lightbeams which in acceptable approximation admit of such description are said to be "quasi-monochromatic." The ideas put forward by Stokes are (as it happens) "robust" in the sense that they extend naturally—and very informatively—to the physics of quasi-monochromatic beams. Suppose, for example, that such a beam were subjected to the beam-analysis protocol described previously; then in place of (20/43) we would have

$$\left.\begin{aligned} S_0 &= \langle J_0\rangle &&= \langle \mathcal{E}_1^2\rangle + \langle \mathcal{E}_2^2\rangle \\ S_1 &= 2\langle J_1\rangle - \langle J_0\rangle &&= \langle \mathcal{E}_1^2\rangle - \langle \mathcal{E}_2^2\rangle \\ S_2 &= 2\langle J_2\rangle - \langle J_0\rangle &&= 2\langle \mathcal{E}_1\mathcal{E}_2\cos\delta\rangle \\ S_3 &= 2\langle J_3\rangle - \langle J_0\rangle &&= 2\langle \mathcal{E}_1\mathcal{E}_2\sin\delta\rangle \end{aligned}\right\} \tag{48}$$

[17] Which, it is useful to notice, might be the *literally* flying spot traced on an oscilloscope screen when

$$\begin{aligned} \text{horizontal input} &= \mathcal{E}_1\cos(\omega t + \delta_1) + \text{noise} \\ \text{vertical input} &= \mathcal{E}_2\cos(\omega t + \delta_2) + \text{noise} \end{aligned}$$

We are, from this point of view, concerned with the description of a simple class of jiggly *Lissajous figures*.

where I have used angle-brackets to denote the time averages of the quantities in question:

$$\langle F\,\rangle \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T F(t)\,dt$$

Under familiar conditions one can, in such contexts, invoke the so-called *ergodic hypothesis* (which is to say, one can—and frequently very usefully—replace time averages by ensemble averages) to write

$$= \int Fp(F)\,dF$$

Returning with this idea to (48), we find ourselves talking about the *statistical properties* of the quasi-monochromatic beam. Evidence that Stokes' parameters are, if not by initial intent, nevertheless wonderfully well-adapted to discussion of the dominant statistical properties of lightbeams emerges from the following little argument: working from (48) we have

$$S_0^2 = \langle \mathcal{E}_1^2\rangle^2 + 2\langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2^2\rangle + \langle \mathcal{E}_2^2\rangle^2 \tag{49.1}$$

$$\begin{aligned} S_1^2+S_2^2+S_3^2 &= \langle \mathcal{E}_1^2\rangle^2 - 2\langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2^2\rangle + \langle \mathcal{E}_2^2\rangle^2 + \langle 2\mathcal{E}_1\mathcal{E}_2\cos\delta\rangle^2 + \langle 2\mathcal{E}_1\mathcal{E}_2\sin\delta\rangle^2\\ &= S_0^2 + 4\big\{\langle \mathcal{E}_1\mathcal{E}_2\cos\delta\rangle^2 + \langle \mathcal{E}_1\mathcal{E}_2\sin\delta\rangle^2 - \langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2^2\rangle\big\} \end{aligned} \tag{49.2}$$

But if x and y are any random variables (however distributed) then from $\langle(\lambda x+y)^2\rangle = \lambda^2\langle x\rangle^2 + 2\lambda\langle xy\rangle + \langle y\rangle^2 \geqslant 0$ (all λ) it follows that in all cases $\langle xy\rangle^2 \leqslant \langle x^2\rangle\langle y^2\rangle$, so we have

$$\begin{aligned} \langle \mathcal{E}_1\mathcal{E}_2\cos\delta\rangle^2 &\leqslant \langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2^2\cos^2\delta\rangle\\ \langle \mathcal{E}_1\mathcal{E}_2\sin\delta\rangle^2 &\leqslant \langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2^2\sin^2\delta\rangle \end{aligned}$$

giving

$$S_1^2+S_2^2+S_3^2 \leq S_0^2 + 4\underbrace{\big\{\langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2(\cos^2\delta+\sin^2\delta)^2\rangle - \langle \mathcal{E}_1^2\rangle\langle \mathcal{E}_2^2\rangle\big\}}_{0}$$

We are led thus to the important inequality

$$(S|S) \equiv S_0^2 - S_1^2 - S_2^2 - S_3^2 \geqslant 0 \tag{50}$$

with equality if (but not only if!) the beam is literally monochromatic. Equivalently, $\cdot \leqslant 1$: the "Poincaré 3-vector" (see again (22)) lies generally *interior to* the Poincaré sphere, and reaches to the surface of the Poincaré sphere only if the beam is, in a fairly evident sense, *statistically equivalent to a monochromatic beam*.

If $\mathcal{E}_1$, $\mathcal{E}_2$ and δ are *statistically independent* random variables then we can in place of (48) write

$$\begin{aligned} S_0 &= \langle J_0\rangle &&= \langle \mathcal{E}_1^2\rangle + \langle \mathcal{E}_2^2\rangle \\ S_1 &= 2\langle J_1\rangle - \langle J_0\rangle &&= \langle \mathcal{E}_1^2\rangle - \langle \mathcal{E}_2^2\rangle \\ S_2 &= 2\langle J_2\rangle - \langle J_0\rangle &&= 2\langle \mathcal{E}_1\rangle\langle \mathcal{E}_2\rangle\langle \cos\delta\rangle \\ S_3 &= 2\langle J_3\rangle - \langle J_0\rangle &&= 2\langle \mathcal{E}_1\rangle\langle \mathcal{E}_2\rangle\langle \sin\delta\rangle \end{aligned}$$

If, moreover, all δ-values are equally likely, then $\langle\cos\delta\rangle = \langle\sin\delta\rangle = 0$, and we have $S_2 = S_3 = 0$. If, moreover, $\langle\mathcal{E}_1\rangle = \langle\mathcal{E}_2\rangle$ then $S_1 = 0$. The resulting beam

$$\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} \quad \text{is said to be } \textit{unpolarized}\text{:} \quad = \mathbf{0}$$

It becomes on this basis natural to introduce the

$$\text{“degree of polarization”}\ P \equiv \frac{\sqrt{S_1^2+S_2^2+S_3^2}}{S_0} = |\,| \tag{51}$$

and to write

$$\begin{pmatrix} S_0\\S_1\\S_2\\S_3 \end{pmatrix} = \begin{pmatrix} PS_0\\S_1\\S_2\\S_3 \end{pmatrix} + \begin{pmatrix} (1-P)S_0\\0\\0\\0 \end{pmatrix}$$

$$= \text{polarized component} \quad + \quad \text{unpolarized component}$$

When an unpolarized beam is presented to (for example) the linear polarizer of (35.1) one obtains

$$\begin{pmatrix} S_0\\S_1\\S_2\\S_3 \end{pmatrix}_{\text{in}} \xrightarrow[\text{linear polarizer at } 0^\circ]{} \begin{pmatrix} S_0\\S_1\\S_2\\S_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} S_0/2\\S_0/2\\0\\0 \end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & 0 & 0\\ \frac12 & \frac12 & 0 & 0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} S_0\\0\\0\\0 \end{pmatrix}_{\text{in}}$$

$P_{\text{in}} = 0$ (the input beam is unpolarized) but $P_{\text{out}} = 1$: the output beam is 100% polarized. Evidently the Muller calculus shares the ”robustness” of the Stokes representation upon which it is based.

We are in position now to appreciate the import of Stokes’

> **Principle of Optical Equivalence**: Lightbeams with identical Stokes parameters are “equivalent” in the sense that they interact identically with devices which detect or alter the intensity and/or polarizational state of the incident beam.

and the depth of his insight into the physics of light. But one does not say of objects that they are, in designated respects, "equivalent" unless there exist other respects—whether overt or covert—in which they are at the same time inequivalent; implicit in the formulation of Stokes' principle is an assertion that physical light beams possess properties beyond those to which the Stokes parameters allude, properties to which photometer-like devices are insensitive. There are many ways to render a page gray with featureless squiggles, many ways to assemble an unpolarized light beam. What such beams, such statistical assemblages share is, according to (48), not "identity" but only the property that a certain quartet of numbers arising from their low-order moments and correlation coefficients are equi-valued. Nor is this remark special to unpolarized beams; it pertains as well to beams in general: monochromaticity *implies but is not implied by* $P = 1$. The optical situation here brought to light is reminiscent of that encountered in (for example) the dynamics of rigid bodies, where the 0^{th}, 1^{st} and 2^{nd} moments

$$
\begin{aligned}
M &= \int \mu(\mathbf{x})\, d^3x && \text{total mass}\\
\mathbf{X} &= \tfrac{1}{M}\int \mathbf{x}\,\mu(\mathbf{x})\, d^3x && \text{center of mass}\\
I_{jk} &= \int (x_j - X_j)(x_k - X_k)\,\mu(\mathbf{x})\, d^3x && \text{moment of inertia matrix}
\end{aligned}
$$

but not the higher moments of the mass distribution $\mu(\mathbf{x})$ enter into the equations of motion; the latter come into play only to the extent that the rigid body moves responsively to the higher derivative structure of some ambient field. So it is in optics. We have, in effect, been alerted by Stokes to the existence of a "statistical optics"—to the possibility that instruments (more subtle in their action than photometers) might be devised which are sensitive to higher moments of an incident optical beam. And we have been alerted to the possible existence and potential usefulness of an *ascending hierarchy* of "higher order analogs" of the parameters which bear Stokes' name, and which do in all events serve to capture the dominant statistical properties of optical beams. Examination of the literature[18] shows all those expectations to be borne out by fairly recent developments. It becomes interesting in the light of these remarks to recall the title of the paper in which the Stokes parameters were first described: "On the composition and resolution of streams of polarized light from different sources" (Trans. Camb. Phil. Soc. **9**, 399 (1852)).

6. The Jones calculus. The material developed in recent sections is associated primarily with the names of Stokes and Muller. I turn now to review of work associated with the names of H. Poincaré (whose contributions to the field were

[18] See, for example, E. L. O'Neill, *Introduction to Statistical Optics* (1963); J. W. Simmons & M. J. Guttmann, *States, Waves and Photons: A Modern Introduction to Light* (1970); C. Brosseau, *Fundamentals of Polarized Light: A Statistical Optics Approach* (1998).

forty years subsequent to those of Stokes) and of R. Clark Jones (whose work, performed while an undergraduate employed in the laboratory of Edwin Land, was contemporaneous with that of Muller)—work notable for the persistent intrusion of $i = \sqrt{-1}$, and which serves, both algebraically and analytically, to deepen our understanding of the material already in hand. In a sense, we will for the most part be decanting old wine into elegant new bottles. But quite apart from the æsthentic satisfaction to be derived from such activity, it will be found to cast old physical relationships in interesting new light, to give rise to new computational techniques, and—which for us is the ultimate point of this whole exercise—to establish direct contact with concepts and methods basic to fields seemingly quite remote from optics. I look to Jones and Poincaré in reversed historical sequence, which is to say: in order of descending mathematical depth. Nothing that Jones had to say would, I think, have been news—fifty years earlier—to Poincaré.

In view of (2) it is entirely natural to introduce

$$\boldsymbol{\mathcal{E}}(t) = \boldsymbol{\mathcal{E}} \cdot e^{i\omega t} \tag{52.1}$$

where

$$\boldsymbol{\mathcal{E}} \equiv \begin{pmatrix} \mathcal{E}_1 e^{i\delta_1} \\ \mathcal{E}_2 e^{i\delta_2} \end{pmatrix} = \mathcal{E}_1 e^{i\delta_1}\mathbf{e}_1 + \mathcal{E}_2 e^{i\delta_1}\mathbf{e}_2 \tag{52.2}$$

defines the so-called "Jones vector" of the monochromatic beam in question. One recovers the physical wave (2) by extraction of the real part of the complex construction (52); here as generally, the "complexification trick" owes its success largely to the fact that

$$\text{real part of superposition} \;=\; \text{superposition of real parts}$$

The Stokes parameters (20) are, however, *quadratic* in the field amplitudes, and of course

$$\text{square of sum} \;\neq\; \text{sum of squares}$$

But if we write

$$\boldsymbol{\mathcal{E}}^\dagger \equiv \text{conjugate transpose of } \boldsymbol{\mathcal{E}}$$

then we have

$$\left.\begin{aligned}
S_0 &= \boldsymbol{\mathcal{E}}^\dagger\, \mathbb{s}_0\, \boldsymbol{\mathcal{E}} \quad \text{with} \quad \mathbb{s}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
S_1 &= \boldsymbol{\mathcal{E}}^\dagger\, \mathbb{s}_1\, \boldsymbol{\mathcal{E}} \quad \text{with} \quad \mathbb{s}_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
S_2 &= \boldsymbol{\mathcal{E}}^\dagger\, \mathbb{s}_2\, \boldsymbol{\mathcal{E}} \quad \text{with} \quad \mathbb{s}_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
S_3 &= \boldsymbol{\mathcal{E}}^\dagger\, \mathbb{s}_3\, \boldsymbol{\mathcal{E}} \quad \text{with} \quad \mathbb{s}_3 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned}\right\} \tag{53}$$

where the 2×2 matrices $\mathbb{s}_\mu$—familiar[19] to the quantum mechanical world as "Pauli matrices"—are hermitian and possess (amongst others) the following notable properties:

$$\operatorname{tr}\mathbb{s}_0 = 2 \quad \text{but} \quad \operatorname{tr}\mathbb{s}_1 = \operatorname{tr}\mathbb{s}_2 = \operatorname{tr}\mathbb{s}_3 = 0 \tag{54}$$

$$\begin{aligned} \mathbb{s}_0 \cdot \mathbb{s}_0 &= \mathbb{s}_0 \\ \mathbb{s}_0 \cdot \mathbb{s}_j = \mathbb{s}_j \cdot \mathbb{s}_0 &= \mathbb{s}_j \\ \mathbb{s}_j \cdot \mathbb{s}_k &= \delta_{jk}\, \mathbb{s}_0 + i \sum_{l=1}^{3} \epsilon_{jkl}\, \mathbb{s}_l \end{aligned} \tag{55}$$

The $\mathbb{s}$-matrices span the space of 2×2 hermitian matrices in the sense that the most general such matrix can be developed

$$\mathbb{H} = \begin{pmatrix} h^0 + \ h^1 & h^2 - ih^3 \\ h^2 + ih^3 & h^0 - \ h^1 \end{pmatrix} = h^0\, \mathbb{s}_0 + h^1\, \mathbb{s}_1 + h^2\, \mathbb{s}_2 + h^3\, \mathbb{s}_3 \tag{56.1}$$

Moreover

$$h^\mu = \tfrac{1}{2}\operatorname{tr}\left(\mathbb{s}_\mu\, \mathbb{H}\right) \tag{56.2}$$

since it is an implication of (54) and (55) that the $\mathbb{s}$-matrices are *trace-wise orthonormal* in the sense that

$$\tfrac{1}{2}\operatorname{tr}\left(\mathbb{s}_\mu\, \mathbb{s}_\nu\right) = \delta_{\mu\nu} \tag{57}$$

From

$$\det(\mathbb{s}_k - \lambda\, \mathbb{I}) = (\lambda + 1)(\lambda - 1) \quad : \quad k = 1, 2, 3$$

we see that $\mathbb{s}_1$, $\mathbb{s}_2$ and $\mathbb{s}_3$ have identical spectra: $\lambda = \pm 1$. The equations

$$\left.\begin{aligned} \mathbb{s}_1 \mathbf{1}_+ &= +\mathbf{1}_+ \quad \text{with} \quad \mathbf{1}_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbb{s}_1 \mathbf{1}_- &= -\mathbf{1}_- \quad \text{with} \quad \mathbf{1}_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ \mathbb{s}_2 \mathbf{2}_+ &= +\mathbf{2}_+ \quad \text{with} \quad \mathbf{2}_+ \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbb{s}_2 \mathbf{2}_- &= -\mathbf{2}_- \quad \text{with} \quad \mathbf{2}_- \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \\ \mathbb{s}_3 \mathbf{3}_+ &= +\mathbf{3}_+ \quad \text{with} \quad \mathbf{3}_+ \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \mathbb{s}_3 \mathbf{3}_- &= -\mathbf{3}_- \quad \text{with} \quad \mathbf{3}_- \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned}\right\} \tag{58}$$

---

[19] Except notationally: my non-standard BLACKBOARD notation is responsive to an anticipated need to distinguish matrices from the operators of which they provide particular representations. More standardly: Latin indices will range on $\{1, 2, 3\}$, Greek indices on $\{0, 1, 2, 3\}$, and the Einstein summation convention will be in force.

assign detailed meaning to the generic equation $\mathbb{s}_k\mathbf{k}_\pm = \pm\mathbf{k}_\pm$. The vectors $\{\mathbf{k}_+, \mathbf{k}_-\}$ are, for each of the three values assignable to k, orthogonal, and since we have taken the trouble to normalize them they are in fact orthonormal

$$\begin{aligned} \mathbf{k}_+^\dagger\mathbf{k}_+ &= \mathbf{k}_-^\dagger\mathbf{k}_- = 1 \\ \mathbf{k}_+^\dagger\mathbf{k}_- &= \mathbf{k}_-^\dagger\mathbf{k}_+ = 0 \end{aligned} \tag{59}$$

and complete

$$\mathbb{I} = \mathbf{k}_+\mathbf{k}_+^\dagger + \mathbf{k}_-\mathbf{k}_-^\dagger \tag{60}$$

Each of the $\mathbb{s}$-matrices serves, in short, to inscribe on complex 2-space—the vector space in which $\boldsymbol{\mathcal{E}}$-vectors live, and upon which our 2×2 matrices act—an orthonormal basis. Each of the $\mathbb{s}$-matrices is diagonal in its own basis

$$\mathbb{s}_k = \mathbf{k}_+\mathbf{k}_+^\dagger - \mathbf{k}_-\mathbf{k}_-^\dagger \tag{61}$$

The matrices/vectors encountered above refer in a particular representation to a more abstract scheme, which I have now to sketch: the fundamental objects are hermitian *Pauli operators* $\boldsymbol{\sigma}_0 \equiv \mathsf{I}$ and $\boldsymbol{\sigma}_k$ which act on $\mathcal{C}_2$ and conform to algebraic relations (55)

$$\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \delta_{jk}\,\mathsf{I} + i\sum_{l=1}^{3}\epsilon_{jkl}\,\boldsymbol{\sigma}_l \tag{62}$$

which entail $[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2\epsilon_{jkl}\,\boldsymbol{\sigma}_l$. Methods borrowed from the quantum theory of angular momentum (spin) supply representation-independent proof that the operators $\boldsymbol{\sigma}_k$ have identical spectra $\lambda = \pm 1$ (and are therefore traceless). In Dirac's elegant notation we have

$$\begin{aligned} \boldsymbol{\sigma}_k|k,+) &= +|k,+) \\ \boldsymbol{\sigma}_k|k,-) &= -|k,-) \end{aligned} \tag{63}$$

It is *relative to the eigenbasis of* $\boldsymbol{\sigma}_1$ that—tacitly—we worked in the preceding paragraph, and it is by writing

$$|E) = \mathcal{E}_1 e^{i\delta_1}|1,+) + \mathcal{E}_2 e^{i\delta_2}|1,-) \tag{64}$$

that we bind the formalism to its concrete physical interpretation. Abstractly we have the representation-independent statements

$$\left.\begin{aligned} S_0 &= (E|\boldsymbol{\sigma}_0|E) \\ S_1 &= (E|\boldsymbol{\sigma}_1|E) \\ S_2 &= (E|\boldsymbol{\sigma}_2|E) \\ S_3 &= (E|\boldsymbol{\sigma}_3|E) \end{aligned}\right\} \tag{65}$$

which, if $\{|\alpha), |\beta)\}$ comprise an *arbitrary* orthonormal basis, acquire the form

$$S_\mu = \begin{pmatrix} (E|\alpha) \\ (E|\beta) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} (\alpha|\boldsymbol{\sigma}_\mu|\alpha) & (\alpha|\boldsymbol{\sigma}_\mu|\beta) \\ (\beta|\boldsymbol{\sigma}_\mu|\alpha) & (\beta|\boldsymbol{\sigma}_\mu|\beta) \end{pmatrix} \begin{pmatrix} (\alpha|E) \\ (\beta|E) \end{pmatrix} \tag{66}$$

typical of a *specific matrix representation*, and if, more particularly, we set $\{|\alpha) = |1,+), |\beta) = |1,-)\}$ we recover (53):

$$S_\mu = \boldsymbol{\mathcal{E}}^\dagger \, \mathbb{s}_\mu \, \boldsymbol{\mathcal{E}}$$

The utility of the abstract formalism lies in the relative ease with which it permits us to move from one representation to another, which I will illustrate by example. . . but by way of preparation: in our "base representation" (eigenbasis of $\boldsymbol{\sigma}_1$), plucking descriptions of the six vectors $\mathbf{k}_\pm$ from (58), we have

$$\begin{pmatrix} \mathcal{E}_1 e^{i\delta_1} \\ \mathcal{E}_2 e^{i\delta_2} \end{pmatrix} = \begin{cases} \mathbf{1}_+ & \text{when we set } \mathcal{E}_1 = 1, \delta_1 = 0, \mathcal{E}_2 = 0, \delta_2 \text{ arbitrary} \\ \mathbf{1}_- & \text{when we set } \mathcal{E}_1 = 0, \delta_1 \text{ arbitrary}, \mathcal{E}_2 = 1, \delta_2 = 0 \\ & \\ \mathbf{2}_+ & \text{when we set } \mathcal{E}_1 = 1/\sqrt{2}, \delta_1 = 0, \mathcal{E}_2 = 1/\sqrt{2}, \delta_2 = 0 \\ \mathbf{2}_- & \text{when we set } \mathcal{E}_1 = 1/\sqrt{2}, \delta_1 = 0, \mathcal{E}_2 = 1/\sqrt{2}, \delta_2 = \pi \\ & \\ \mathbf{3}_+ & \text{when we set } \mathcal{E}_1 = 1/\sqrt{2}, \delta_1 = 0, \mathcal{E}_2 = 1/\sqrt{2}, \delta_2 = +\pi/2 \\ \mathbf{3}_- & \text{when we set } \mathcal{E}_1 = 1/\sqrt{2}, \delta_1 = 0, \mathcal{E}_2 = 1/\sqrt{2}, \delta_2 = -\pi/2 \end{cases}$$

according to which (by (20))

$$\left.\begin{array}{l} \mathbf{1}_+ \text{ refers in the } \boldsymbol{\sigma}_1\text{-eigenbasis to } \longleftrightarrow \\ \mathbf{1}_- \text{ refers in the } \boldsymbol{\sigma}_1\text{-eigenbasis to } \updownarrow \\ \mathbf{2}_+ \text{ refers in the } \boldsymbol{\sigma}_1\text{-eigenbasis to } \swarrow\!\!\!\nearrow \\ \mathbf{2}_- \text{ refers in the } \boldsymbol{\sigma}_1\text{-eigenbasis to } \nwarrow\!\!\!\searrow \\ \mathbf{3}_+ \text{ refers in the } \boldsymbol{\sigma}_1\text{-eigenbasis to } \circlearrowright \\ \mathbf{3}_- \text{ refers in the } \boldsymbol{\sigma}_1\text{-eigenbasis to } \circlearrowleft \end{array}\right\} \tag{67}$$

By a convention which I find backwards, clockwise circulation $\circlearrowright$ is said by opticians (and by the designer of $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX, who assigned to the symbol $\circlearrowright$ the code `\circlearrowright`) to be "right-handed," and $\circlearrowleft$ to be lefthanded.

To change basis, one proceeds in direct mimicry of standard quantum mechanical practice, writing

$$|E) = \sum |\alpha)(\alpha|E) \tag{68.1}$$

$$(\alpha|E) = \sum (\alpha|\tilde{\alpha})(\tilde{\alpha}|E) \tag{68.2}$$

More concretely,

$$\begin{pmatrix} (\alpha|E) \\ (\beta|E) \end{pmatrix} = \begin{pmatrix} (\alpha|\tilde{\alpha}) & (\alpha|\tilde{\beta}) \\ (\beta|\tilde{\alpha}) & (\beta|\tilde{\beta}) \end{pmatrix} \begin{pmatrix} (\tilde{\alpha}|E) \\ (\tilde{\beta}|E) \end{pmatrix} \tag{69}$$

$$\downarrow$$

$$\boldsymbol{\mathcal{E}} = \mathbb{U}\,\tilde{\boldsymbol{\mathcal{E}}}$$

Suppose, for example, that

$$\begin{aligned}&\big\{|\alpha),|\beta)\big\} \text{ refer, as above, to the } \boldsymbol{\sigma}_1\text{-eigenbasis } \longleftrightarrow, \updownarrow\\&\big\{|\tilde\alpha),|\tilde\beta)\big\} \text{ refer to the } \boldsymbol{\sigma}_3\text{-eigenbasis } \circlearrowright, \circlearrowleft\end{aligned}$$

Then, reading from (58), we have

$$\begin{aligned}|\tilde\alpha) &= |\alpha)(\alpha|\tilde\alpha) + |\beta)(\beta|\tilde\alpha) & |\tilde\beta) &= |\alpha)(\alpha|\tilde\beta) + |\beta)(\beta|\tilde\beta)\\&= \tfrac{1}{\sqrt2}\big\{|\alpha)\cdot 1 + |\beta)\cdot i\big\} & &= \tfrac{1}{\sqrt2}\big\{|\alpha)\cdot 1 - |\beta)\cdot i\big\}\end{aligned}$$

giving

$$\mathbb{U} = \tfrac{1}{\sqrt2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} \tag{70.1}$$

and so obtain

$$\left.\begin{aligned}S_0 &= (E|\boldsymbol{\sigma}_0|E) = \boldsymbol{\mathcal{E}}^\dagger\mathbb{s}_0\boldsymbol{\mathcal{E}} = \tilde{\boldsymbol{\mathcal{E}}}^\dagger\tilde{\mathbb{s}}_0\tilde{\boldsymbol{\mathcal{E}}} \quad\text{with}\quad \tilde{\mathbb{s}}_0 \equiv \mathbb{U}^\dagger\mathbb{s}_0\mathbb{U} = \mathbb{s}_0\\S_1 &= (E|\boldsymbol{\sigma}_1|E) = \boldsymbol{\mathcal{E}}^\dagger\mathbb{s}_1\boldsymbol{\mathcal{E}} = \tilde{\boldsymbol{\mathcal{E}}}^\dagger\tilde{\mathbb{s}}_1\tilde{\boldsymbol{\mathcal{E}}} \quad\text{with}\quad \tilde{\mathbb{s}}_1 \equiv \mathbb{U}^\dagger\mathbb{s}_1\mathbb{U} = \mathbb{s}_2\\S_2 &= (E|\boldsymbol{\sigma}_2|E) = \boldsymbol{\mathcal{E}}^\dagger\mathbb{s}_2\boldsymbol{\mathcal{E}} = \tilde{\boldsymbol{\mathcal{E}}}^\dagger\tilde{\mathbb{s}}_2\tilde{\boldsymbol{\mathcal{E}}} \quad\text{with}\quad \tilde{\mathbb{s}}_2 \equiv \mathbb{U}^\dagger\mathbb{s}_2\mathbb{U} = \mathbb{s}_3\\S_3 &= (E|\boldsymbol{\sigma}_3|E) = \boldsymbol{\mathcal{E}}^\dagger\mathbb{s}_3\boldsymbol{\mathcal{E}} = \tilde{\boldsymbol{\mathcal{E}}}^\dagger\tilde{\mathbb{s}}_3\tilde{\boldsymbol{\mathcal{E}}} \quad\text{with}\quad \tilde{\mathbb{s}}_3 \equiv \mathbb{U}^\dagger\mathbb{s}_3\mathbb{U} = \mathbb{s}_1\end{aligned}\right\} \tag{70.2}$$

and

$$\begin{aligned}\tilde{\boldsymbol{\mathcal{E}}} = \mathbb{U}^\dagger\boldsymbol{\mathcal{E}} &= \tfrac{1}{\sqrt2}\begin{pmatrix}\mathcal{E}_1e^{i\delta_1} - i\mathcal{E}_2e^{i\delta_2}\\ \mathcal{E}_1e^{i\delta_1} + i\mathcal{E}_2e^{i\delta_2}\end{pmatrix}\\&\equiv \begin{pmatrix}\mathcal{A}e^{i\alpha}\\ \mathcal{B}e^{i\beta}\end{pmatrix} \quad:\quad \text{reproduces (39)}\end{aligned} \tag{70.3}$$

So

$$\left.\begin{aligned}S_0 &= \mathcal{A}^2 + \mathcal{B}^2\\S_1 &= 2\mathcal{A}\mathcal{B}\cos\gamma \quad:\quad \gamma \equiv \beta - \alpha\\S_2 &= 2\mathcal{A}\mathcal{B}\sin\gamma\\S_3 &= \mathcal{A}^2 - \mathcal{B}^2\end{aligned}\right\} \tag{70.4}$$

We have in (70.3) achieved a wonderfully succinct rendition of (39), and have in (70.4) recovered (41), but by a more transparent argument, and with a lot less labor.

The matrix $\mathbb{U}$ is unprepossessing on its face, but has in fact an interesting story to tell. The point to notice is that $\det\mathbb{U} = -i$ so $\mathbb{U}$, though unitary, is not unimodular; it becomes, in this light, natural[20] to write

$$\mathbb{U} = \sqrt{-i}\cdot\underbrace{\tfrac12(1+i)\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}}_{\text{unimodular factor}}$$

[20] Observe that $\sqrt{-i} = e^{-i\frac{\pi}{4}}$ and $e^{+i\frac{\pi}{4}} = \frac{1}{\sqrt2}(1+i)$.

and to notice that

$$\begin{aligned}\text{unimodular factor} &= \tfrac{1}{2}\begin{pmatrix}1+i & 1+i\\ -1+i & 1-i\end{pmatrix}\\ &= \tfrac{1}{2}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right\}\\ &= \tfrac{1}{2}\mathbb{S}_0 + i\tfrac{\sqrt{3}}{2}\left\{\tfrac{1}{\sqrt{3}}\mathbb{S}_1 + \tfrac{1}{\sqrt{3}}\mathbb{S}_2 + \tfrac{1}{\sqrt{3}}\mathbb{S}_3\right\}\\ &= \exp\left\{i\tfrac{\pi}{3}\big[\lambda_1\mathbb{S}_1 + \lambda_2\mathbb{S}_2 + \lambda_3\mathbb{S}_3\big]\right\} \quad \text{with} \quad \boldsymbol{\lambda} \equiv \begin{pmatrix}1\\1\\1\end{pmatrix}\end{aligned} \tag{71}$$

is precisely the element of $SU(2)$ which is associated[21] with the element of $O(3)$ that refers to *rotation through* $\frac{2\pi}{3}$ *radians* $(120°)$ *about the unit vector* $\boldsymbol{\lambda}$. The induced action of $\mathbb{U}$ in 3-dimensional -space can be described

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix} \;;\; \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix} \;;\; \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}$$

as foreshadowed at (70.2).

Except for the fact that no condition of the form $(E|E) = 1$ is operative, equations (65) very much resemble the equations which in quantum mechanics are used to compute "expectation values." It becomes natural in the light of this remark to notice that (65) admit of this alternative formulation:

$$S_\mu = \mathrm{tr}\,(\boldsymbol{\sigma}_\mu \mathbf{b}) \quad \text{where} \quad \mathbf{b} \equiv |E)(E| \tag{72}$$

Since the "beam operator" $\mathbf{b}$ is hermitian on $\mathcal{C}_2$ it can be developed as a linear combination of $\boldsymbol{\sigma}$-operators, and from

$$S_\mu = \mathrm{tr}\,(\boldsymbol{\sigma}_\mu \sum_\nu b_\nu \boldsymbol{\sigma}_\nu) = \sum_\nu b_\nu \mathrm{tr}\,(\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\nu) = 2\sum_\nu b_\nu \delta_{\mu\nu} = 2b_\mu$$

we learn that Stokes' parameters are essentially the "coordinates" of $\mathbf{b}$; we have the association

$$|E) \longleftrightarrow \mathbf{b} = \tfrac{1}{2}\big\{S_0\boldsymbol{\sigma}_0 + S_1\boldsymbol{\sigma}_1 + S_2\boldsymbol{\sigma}_2 + S_3\boldsymbol{\sigma}_3\big\} \tag{73}$$

The Jones vector $|E)$ and the beam operator $\mathbf{b}$ are equivalent in the sense that, for all hermitian "observables" A,

$$\langle \mathsf{A} \rangle_E = (E|\mathsf{A}|E) = \mathrm{tr}\,(\mathsf{A}\,\mathbf{b}) \tag{74}$$

but they are inequivalent in this weak respect:

$$\mathbf{b} \text{ is invariant under } |E) \longrightarrow e^{i\alpha}|E)$$

[21] See, for example, CLASSICAL DYNAMICS, (1964), Chapter 1, p. 118.

$\mathbf{b}$ is, in short, sensitive to relative phase $\delta \equiv \delta_2 - \delta_1$ but insensitive to the absolute phase of the physical wave—as also, for that matter, is $\langle \mathsf{A} \rangle_E$. In representation with respect to any specified coordinate system the "beam operator" becomes the "beam matrix:"

$$\mathbf{b} \text{ becomes } \tfrac{1}{2}\begin{pmatrix} S_0 + \ S_1 & S_2 - iS_3 \\ S_2 + iS_3 & S_0 - \ S_1 \end{pmatrix} \text{ in standard representation} \tag{75}$$

where—here as always—by "standard representation" I presume selection of the $\boldsymbol{\sigma}_1$-eigenbasis; I refer, in other words, to the $\longleftrightarrow\updownarrow$ "linear polarization" representation, with respect to which the representatives of the $\boldsymbol{\sigma}$-operators become the Pauli matrices (53). The following equations are immediate in the standard representation

$$\operatorname{tr}\mathbf{b} = S_0 \tag{76.1}$$

$$\det\mathbf{b} = S_0^2 - S_1^2 - S_2^2 - S_3^2 \tag{76.2}$$

but hold in *all* unitarily equivalent representations, and it is for that reason that the expressions on the left become meaningful as written.

Insight into the nature of the density matrix, in its present manifestation, can be obtained as follows: from $|E)(E| \cdot |E)(E| = |E)S_0(E|$ we conclude that

$$\mathbf{b}^2 = S_0\mathbf{b} \tag{77}$$

It is, on the other hand, an implication of the Cayley-Hamilton theorem that $\mathbf{b}^2 - (\operatorname{tr}\mathbf{b})\mathbf{b} + (\det\mathbf{b})\,\mathsf{I} = \mathbf{0}$ which by (76) reads

$$\mathbf{b}^2 = S_0\mathbf{b} - (S_0^2 - S_1^2 - S_2^2 - S_3^2)\,\mathsf{I} \tag{78}$$

Comparison of (77) with (78) returns the familiar information that (for monochromatic beams)

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = 0$$

With the consistency of (77) and (78) thus established, we discover that

$$\mathsf{P} \equiv \tfrac{1}{S_0}\mathbf{b} \text{ is a projection matrix} : \mathsf{P}^2 = \mathsf{P} \tag{79}$$

Explicitly

$$\mathsf{P} = |\hat{E})(\hat{E}| \quad \text{where } |\hat{E}) \equiv \tfrac{1}{\sqrt{S_0}}|E) \text{ is normalized} : (\hat{E}|\hat{E}) = 1 \tag{80.1}$$

$$= \tfrac{1}{2}\big\{\boldsymbol{\sigma}_0 + s_1\boldsymbol{\sigma}_1 + s_2\boldsymbol{\sigma}_2 + s_3\boldsymbol{\sigma}_3\big\} \tag{80.2}$$

where (as previously) $s_k \equiv S_k/S_0$. From $\det(\,\mathsf{P} - \lambda\,\mathsf{I}\,) = \lambda(\lambda - 1)$ we learn that P projects onto a 1-dimensional subspace of $\mathcal{C}_2$ (i.e., that $\mathbf{p}$ projects *onto a ray*). Evidently

$$\mathsf{P}_\perp \equiv \mathsf{I} - \mathsf{P} \tag{81.1}$$

$$= \tfrac{1}{2}\{\boldsymbol{\sigma}_0 - s_1\boldsymbol{\sigma}_1 - s_2\boldsymbol{\sigma}_2 - s_3\boldsymbol{\sigma}_3\} \tag{81.2}$$

projects onto the orthogonal complement of that ray, and in $\{\mathsf{P},\mathsf{P}_\perp\}$ we have a complete ($\mathsf{P}+\mathsf{P}_\perp=\mathsf{I}$) orthogonal ($\mathsf{P}\cdot\mathsf{P}_\perp=\mathbf{0}$) set of projection operators ($\mathsf{P}^2=\mathsf{P}$ and $\mathsf{P}_\perp^2=\mathsf{P}_\perp$). As to the identity of the ray onto which P projects, it is clear from (80.1) that $\mathsf{P}|E)=|E)$: P projects onto the ray which contains (and is defined by) the Jones vector $|E)$. If $|\hat{E})_\perp$ denotes the unit vector normal to $|\hat{E})$[22]

$$(\hat{E}|\hat{E})={}_\perp(\hat{E}|\hat{E})_\perp=1 \quad\text{and}\quad (\hat{E}|\hat{E})_\perp=0$$

then

$$\begin{aligned}\mathsf{P}\,|\hat{E})&=|\hat{E}) &\qquad \mathsf{P}\,|\hat{E})_\perp&=0\\ \mathsf{P}_\perp|\hat{E})&=0 &\qquad \mathsf{P}_\perp|\hat{E})_\perp&=|\hat{E})_\perp\end{aligned}$$

While the assembly ($|E)\longrightarrow\mathbf{b}$) of $\mathbf{b}$ for given $|E)$ is by (72) made trivial, the reverse procedure ($|E)\longleftarrow\mathbf{b}$) is not quite trivial, but (as we have just learned) soluble as follows: (i) compute $S_0=\mathrm{tr}\,\mathbf{b}$; ($ii$) construct $\mathsf{P}\equiv\mathbf{b}/S_0$; ($iii$) construct the eigenvector $|E)$ defined by $\mathsf{P}\,|E)=|E)$ and (iv) impose the normalization condition $(E|E)=S_0$. The resulting $|E)$ is determined *to within an overall phase factor*.

It is interesting to note that, while $|E)$ and its complement $|E)_\perp$ are *orthogonal*, they give rise in Stokes' representation to -vectors which are (compare (80.2) with (81.2)) *anti-parallel*; such is the variety of the languages available to us when we wish to refer to states of "opposite polarization."

To describe the beam-modification properties of linear devices, Jones found it natural to write

$$|E)_{\mathrm{in}}\longrightarrow|E)_{\mathrm{out}}=\mathbf{J}\,|E)_{\mathrm{in}} \tag{82}$$

Equivalent (except of the abandonment of an overall phase factor) is

$$\mathbf{b}_{\mathrm{in}}\longrightarrow\mathbf{b}_{\mathrm{out}}=\mathbf{J}\,\mathbf{b}_{\mathrm{in}}\,\mathbf{J}^\dagger \tag{83}$$

where $\mathbf{J}^\dagger$ is the adjoint of $\mathbf{J}$. Equations (82) and (83) are to be compared to (44); clearly, the Jones calculus is situated in complex 2-space, and speaks of the blackbox adventures of the *physical* wave itself, while the Muller calculus is situated in a real 4-space, and speaks only/directly of the the *observable properties* of the physical wave. Returning with (82) to (65), we obtain

$$\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}_{\mathrm{in}}=\begin{pmatrix}(E|\boldsymbol{\sigma}_0|E)\\(E|\boldsymbol{\sigma}_1|E)\\(E|\boldsymbol{\sigma}_2|E)\\(E|\boldsymbol{\sigma}_3|E)\end{pmatrix}\longrightarrow\begin{pmatrix}(E|\mathbf{J}^\dagger\boldsymbol{\sigma}_0\mathbf{J}|E)\\(E|\mathbf{J}^\dagger\boldsymbol{\sigma}_1\mathbf{J}|E)\\(E|\mathbf{J}^\dagger\boldsymbol{\sigma}_2\mathbf{J}|E)\\(E|\mathbf{J}^\dagger\boldsymbol{\sigma}_3\mathbf{J}|E)\end{pmatrix}=\begin{pmatrix}S_0\\S_1\\S_2\\S_3\end{pmatrix}_{\mathrm{out}} \tag{84}$$

The hermiticity of $\boldsymbol{\sigma}_\mu$ implies that of $\mathbf{J}^\dagger\boldsymbol{\sigma}_\mu\mathbf{J}$, so we are assured of the existence of real numbers $M_{\mu\nu}$ such that

$$\mathbf{J}^\dagger\boldsymbol{\sigma}_\mu\mathbf{J}=\sum_{\nu=0}^{3}M_{\mu\nu}\boldsymbol{\sigma}_\nu \tag{85.1}$$

[22] I say *the* but mean *a*; such things are defined only to within a phase factor.

—numbers which are given in fact by

$$M_{\mu\nu} = \tfrac{1}{2}\mathrm{tr}\,(\mathbf{J}^\dagger\boldsymbol{\sigma}_\mu\mathbf{J}\boldsymbol{\sigma}_\nu) \tag{85.2}$$

Thus do we recover precisely (44), from which the Muller calculus radiates. In a manner of speaking

$$\text{Jones calculus} = \sqrt{\text{Muller calculus}}$$

but while it is (as we have just witnessed) typically easy to square things, the extraction of roots frequently demands a high order or ingenuity; it is only by force of fairly deep analysis that one is led from Muller-like structures to the invention of—or to an appreciation of the naturalness of—Jones-like structures.

We have observed already that to catalog the properties of the Muller matrices $\mathbb{M}$ is, in effect, to construct a "general theory of linear optical devices." The same can be said of the Jones operators $\mathbf{J}$ (which in representation become 2×2 Jones matrices $\mathbb{J}$). But while those distinct exercises lead necessarily to the same ultimate conclusions, they differ markedly in their details. Suppose, by way of illustration, that the device in question is *non-absorptive*: to impose such a condition is, by (84), is to require that $(S_0)_{\text{out}} = (E|\mathbf{J}^\dagger\mathbf{J}|E) = (E|E) = (S_0)_{\text{in}}$ for all $|E)$, and amounts therefore to a requirement that $\mathbf{J}$ be unitary:

$$\mathbf{J} = e^{i(\text{phase})}\cdot e^{i(\lambda_1\boldsymbol{\sigma}_1+\lambda_2\boldsymbol{\sigma}_2+\lambda_3\boldsymbol{\sigma}_3)}$$

Returning with this information to (85) one at length recovers the previously encountered

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix}$$

as an instance of the famous $SU(2)$ representation—the "spinor representation" —of $O(3)$. Or consider the class of devices called "polarizers." The action of such devices is, as we have observed, projective (in which fundamental respect they serve to *model the action of all quantum mechanical measurement devices*!). If (p_1, p_2, p_3) are components of a unit 3-vector, the Stokes vector descriptive of the output of the polarizer, then it was seen already at (80.2) that the associated Jones operator can be described

$$\mathbf{J}_{\text{polarizer}} = \tfrac{1}{2}\big\{\boldsymbol{\sigma}_0 + p_1\boldsymbol{\sigma}_1 + p_2\boldsymbol{\sigma}_2 + p_3\boldsymbol{\sigma}_3\big\}$$

This simple result could—with labor—be used in conjunction with (85) to construct, in identical generality, a discription of the Muller matrices associated with such devices. As the preceeding examples suggest, the Jones calculus is in many applications notable for its computational efficiency.[23]

[23] For further discussion of the practical particulars of the subject, see (for example) Chapter 10 of Edward Collett's *Polarized Light: Fundamentals & Applications* (1993).

Jones' work will fill quantum physicists—as it must have already in 1941—with a distinct sense of *déjà vu*,[24] for the mathematical ideas which he pressed into optical service have/had for a long time been standard to the quantum mechanics of 2-state systems (spin systems). But that's alright; the recognition, whenever it occurs, that seemingly distinct subjects are (like quantum field theory and statistical mechanics) structurally similar is invariably empowering to students of *both* subjects.

7. Poincaré's contribution. The alternative formalisms sketched thus far are equivalent only insofar as they overlap. And the enhanced "computational efficiency" we have achieved has been purchased at a price: by progressive abandonment physical detail—which has been the pattern of our progress

$$|E) \xrightarrow[\text{loss of absolute phase data}]{} \mathbf{b}$$

and

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \xrightarrow[\text{loss of intensity data}]{} \equiv \begin{pmatrix} S_1/S_0 \\ S_2/S_0 \\ S_3/S_0 \end{pmatrix} \equiv \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

—we find ourselves speaking ever more sharply about less and less. So the trend now continues...though, it might be argued, in service more of sheer elegance than of analytical power.

The Poincaré sphere—which lives in 3-dimensional Stokes space and is defined by the equation $\cdot = 1$—is useful when one has interest only in (the action of sequenced devices upon) the *figure* (orientation, shape and chirality) of the electrical ellipse; when one has, that is to say,

- no interest in the scale/size of the ellipse, and
- no interest in partial polarization (represented by points *interior* to the "Poincaré ball").

But within that limited context it is *quite* useful; its elegant ramifications radiate in several directions, of which I look now to only one.

By stereographic projection from the "north pole" of the Poincaré sphere (i.e., from the point $(0,0,1)$, which is the circular polarization point $\circlearrowright$) onto

[24] Perusal of his papers suggests that is was, however, not *déjà vu* to Jones. In "A new calculus for the treatment of optical systems, Part I: Description and discussion of the calculus," J. Opt. Soc. Am. **31**, 488 (1941) he does cite quantum texts by E. C. Kemble (1937) and by V. Rojansky (1938) as sources for his matrix theory, but nowhere in eight papers spread over fifteen years does he mention a Pauli matrix or a spinor, or show evidence of any close knowledge of quantum mechanics. The work of Stokes, Soleillet and Mueller is mentioned for the first (and only) time in "Part V: A more general formulation, & description of another calculus." J. Opt. Soc. Am. **37**, 107 (1947), and his solitary reference to Poincaré (Part II) is similarly cursory. Jones' papers—all written from the Polaroid Corporation—impress me as the work of an exceptionally inventive young engineer with an aversion to the literature.

the equatorial plane (defined by the equation $S_3 = 0$) one sets up a one-to-one correspondence between points on the surface of the sphere and points $\{u, v\}$ on the plane, which $\boxed{\text{STEP ONE}}$ Poincaré proceeds to associate with the *complex* plane. Working from Figure 4, we are led quickly to the "Poincaré polarization parameters"

$$\left.\begin{aligned} u &\equiv \frac{s_1}{1-s_3} \\ v &\equiv \frac{s_2}{1-s_3} \end{aligned}\right\} \tag{86}$$

Inversely

$$\left.\begin{aligned} s_1 &= \frac{2u}{u^2+v^2+1} \\ s_2 &= \frac{2v}{u^2+v^2+1} \\ s_3 &= \frac{u^2+v^2-1}{u^2+v^2+1} \end{aligned}\right\} \tag{87}$$

With Poincaré we now assemble the complex variable

$$z = u + iv = \frac{s_1 + is_2}{1 - s_3} \equiv z() \tag{88}$$

in terms of which (87) can be written

$$\left.\begin{aligned} s_1 + is_2 &= \frac{2z}{z^*z+1} \\ s_1 - is_2 &= \frac{2z^*}{z^*z+1} \\ s_3 &= \frac{z^*z-1}{z^*z+1} \end{aligned}\right\} \tag{89}$$

We notice it to be an implication of (89) that

$$\begin{aligned} s_1^2 + s_2^2 + s_3^2 &= (s_1+is_2)(s_1-is_2) + s_3^2 \\ &= \frac{4z^*z + (z^*z-1)^2}{(z^*z+1)^2} \\ &= 1 \text{ automatically/inescapably} \end{aligned} \tag{90}$$

And, since we have $(s_1 + is_2)(s_1 - is_2) = (1 + s_3)(1 - s_3)$ by (90), it is an implication of (88) that

$$\begin{aligned} z(-) = -\frac{s_1+is_2}{1+s_3} &= -\frac{1-s_3}{(s_1+is_2)(s_1-is_2)}(s_1+is_2) \\ &= -\frac{1-s_3}{s_1-is_2} \\ &= -\frac{1}{z^*()} \end{aligned} \tag{91}$$

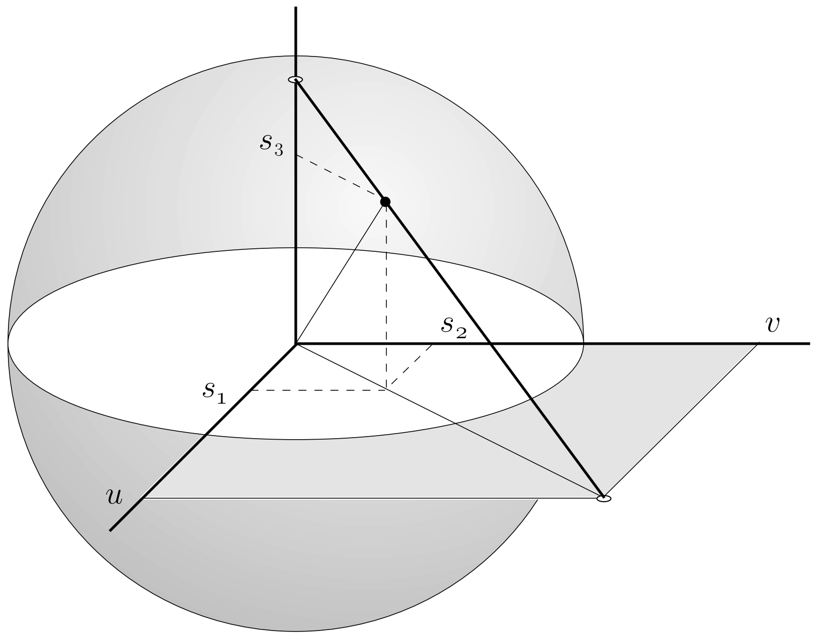

FIGURE 4: *Stereographic projection of the Poincaré sphere onto the equatorial plane gives (by similar triangle arguments)*

$$\frac{1}{\sqrt{u^2+v^2}} = \frac{s_3}{\sqrt{u^2+v^2} - \sqrt{s_1^2+s_2^2}}$$
$$\frac{\sqrt{s_1^2+s_2^2}}{\sqrt{u^2+v^2}} = \frac{s_1}{u} = \frac{s_2}{v}$$

from which follow (86).

So we have in

$$\longrightarrow - \quad\iff\quad |E) \longrightarrow |E)_\perp \quad\iff\quad z \longrightarrow -1/z^*$$

three distinct but equivalent characterizations of what it means to reverse (in the sense " make opposite") the polarization of a beam; we have *associated beam polarizations with points on the z-plane*, and have done so in such a way that opposite polarizations are associated with points which are, in a sense standard to geometry,[25] "reciprocal." Moreover, we have in (85) an invitation to represent beam transformations ${}_{\text{old}} \longrightarrow {}_{\text{new}}$ as automorphisms $z_{\text{old}} \longrightarrow z_{\text{new}}$ of the complex plane. Thus did Poincaré establish contact with a subject which

[25] See Chapter 17, §3 in W. C. Graustein, *Introduction to Higher Geometry* (1930). The point to notice is that $z = re^{i\varphi}$ and $\frac{1}{z^*} = \frac{1}{r}e^{i\varphi}$ lie on a line containing the origin (while $\frac{1}{z} = \frac{1}{r}e^{-i\varphi}$ lies generally *off* that z-ray).

was especially close to his heart, a subject concerning which he was, in his time, a leading expert. Well known to him was the remarkable fact[26] that

> The most general analytic (or conformal) transformation $z \longrightarrow z' = f(z)$ which maps the plane one-to-one into itself is the "linear fractional transformation"[27]

$$z' = \frac{az+b}{cz+d} \tag{92}$$

With Poincaré we next $\boxed{\text{STEP TWO}}$ introduce what in projective geometry would be called "homogeneous coordinates:" complex numbers Z_1^* and Z_2^* whose *ratio* is z:

$$z = \frac{kZ_1^*}{kZ_2^*} \quad : \quad k \text{ arbitrary} \tag{93}$$

> **Remark**: Were we, at this point, to exercise our option to require
>
> $$Z_1^* Z_1 + Z_2^* Z_2 \equiv X_1^2 + Y_1^2 + X_2^2 + Y_2^2 = 1$$
>
> then we would, in effect, have achieved a mapping from the unit sphere S^3 in 4-space to the unit (Poincaré) sphere S^2 in 3-space; we have encountered the historic first instance of a *Hopf mapping*, which in the general case achieves
>
> $$S^{2n-1} \mapsto S^n$$
>
> and in 1931 acquired fundamental importance in algebraic topology.[28]

By slight rearrangement of (89) we have

$$\left.\begin{aligned} s_1 &= \,\frac{z^*+z}{z^*z+1} \\ s_2 &= i\,\frac{z^*-z}{z^*z+1} \\ s_3 &= \,\frac{z^*z-1}{z^*z+1} \end{aligned}\right\} \tag{94}$$

which in the notation introduced at (96) becomes

$$\left.\begin{aligned} s_1 &= \frac{S_1}{S_0} = \,\frac{Z_1^*Z_2 + Z_2^*Z_1}{Z_1^*Z_1 + Z_2^*Z_2} \\ s_2 &= \frac{S_2}{S_0} = -i\,\frac{Z_1^*Z_2 - Z_2^*Z_1}{Z_1^*Z_1 + Z_2^*Z_2} \\ s_3 &= \frac{S_3}{S_0} = \,\frac{Z_1^*Z_1 - Z_2^*Z_2}{Z_1^*Z_1 + Z_2^*Z_2} \end{aligned}\right\} \tag{95}$$

[26] See, for example, L. R. Ford, *Automorphic Functions* (1951), p. 2.

[27] Such transformations are sometimes said to be "bilinear," and sometimes called "Möbius transformations."

[28] For the simplest discussion I have been able to discover, see Chapter 12 of C. T. C. Wall, *A Geometric Introduction to Topology*, (1972).

Finally $\boxed{\text{STEP THREE}}$ we introduce the complex 2-vector

$$\boldsymbol{\mathcal{Z}} \equiv \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \tag{96}$$

and observe that (98) can be notated

$$\left.\begin{aligned} s_1 &= \frac{\boldsymbol{\mathcal{Z}}^\dagger \mathbb{s}_2 \boldsymbol{\mathcal{Z}}}{\boldsymbol{\mathcal{Z}}^\dagger \mathbb{s}_0 \boldsymbol{\mathcal{Z}}} \\ s_2 &= \frac{\boldsymbol{\mathcal{Z}}^\dagger \mathbb{s}_3 \boldsymbol{\mathcal{Z}}}{\boldsymbol{\mathcal{Z}}^\dagger \mathbb{s}_0 \boldsymbol{\mathcal{Z}}} \\ s_3 &= \frac{\boldsymbol{\mathcal{Z}}^\dagger \mathbb{s}_1 \boldsymbol{\mathcal{Z}}}{\boldsymbol{\mathcal{Z}}^\dagger \mathbb{s}_0 \boldsymbol{\mathcal{Z}}} \end{aligned}\right\} \tag{97}$$

This is a pretty result,[29] reminiscent of (53) except for the circumstance that the Pauli matrices are now scrambled. They are, however, scambled in a familiar way (see again (70.2)), and for an intelligible reason: At (53) we were working in the $\longleftrightarrow\!\updownarrow$ basis (eigenbasis of $\mathbb{s}_1$), while Poincaré—when at $\boxed{\text{STEP ONE}}$ he projected from the north $\circlearrowright$ pole—has tacitly elected to work in the $\circlearrowright\circlearrowleft$ basis (eigenbasis of $\mathbb{s}_3$).

At (82) we obtained a representation-independent description of what Jones has to say about the beam-modification properties of a linear device. In the $\longleftrightarrow\!\updownarrow$ representation that equation assumes the form

$$\begin{pmatrix} \mathcal{E}_1 e^{i\delta_1} \\ \mathcal{E}_2 e^{i\delta_2} \end{pmatrix}_{\text{out}} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 e^{i\delta_1} \\ \mathcal{E}_2 e^{i\delta_2} \end{pmatrix}_{\text{in}} \tag{98}$$

Poincaré's construction alerts us to the news that if we write this appropriately unscrambled variant of (94)

$$\left.\begin{aligned} s_1 &= \frac{w^* w - 1}{w^* w + 1} \\ s_2 &= \frac{w^* + w}{w^* w + 1} \\ s_3 &= i\,\frac{w^* - w}{w^* w + 1} \end{aligned}\right\} \tag{99}$$

and define

$$w \equiv \left[\frac{\mathcal{E}_1 e^{i\delta_1}}{\mathcal{E}_2 e^{i\delta_2}}\right]^* = \tfrac{\mathcal{E}_1}{\mathcal{E}_2} e^{i\delta}$$

then we recover (20):

$$\begin{aligned} s_1 &= \frac{\mathcal{E}_1^2 - \mathcal{E}_2^2}{\mathcal{E}_1^2 + \mathcal{E}_2^2} \\ s_2 &= \frac{2\mathcal{E}_1\mathcal{E}_2 \cos\delta}{\mathcal{E}_1^2 + \mathcal{E}_2^2} \\ s_3 &= \frac{2\mathcal{E}_1\mathcal{E}_2 \sin\delta}{\mathcal{E}_1^2 + \mathcal{E}_2^2} \end{aligned}$$

[29] It was to make it so that the unexpected $*$'s were introduced into (93).

8. Optimal superposition of monochromatic beams: Pancharatnam's theorem. To develop the characteristics S_μ of the monochromatic beam $A \oplus B$ which results from physical superposition of

- monochromatic beam A, with characteristics $S_{A\mu}$, and
- monochromatic beam B of the same color, with characteristics $S_{B\mu}$

one must take into account a physical circumstance concerning which the Stokes parameters $\{S_{A\mu}, S_{B\mu}\}$ convey no information: one must take into account the *relative phase* of the constituent beams. How is this to be done?

To describe the composite beam we might write

$$|E) = |E_A) + e^{-i\phi}|E_B) \tag{100}$$

Then

$$\begin{aligned} S_\mu &= (E|\boldsymbol{\sigma}_\mu|E) \\ &= S_{A\mu} + S_{B\mu} + \big\{(E_A|\boldsymbol{\sigma}_\mu|E_B)e^{-i\phi} + \text{complex conjugate}\big\} \\ &\qquad\qquad (E_A|\boldsymbol{\sigma}_\mu|E_B) \equiv R_\mu e^{i\theta_\mu} \ \text{ in polar representation} \\ &= S_{A\mu} + S_{B\mu} + 2R_\mu \cos(\theta_\mu - \phi) \end{aligned} \tag{101}$$

In particular,

$$\text{intensity of composite beam} \sim S_0 = S_{A0} + S_{B0} + 2R_0 \cos(\theta_0 - \phi) \tag{102}$$

will be maximal at $\phi = \theta_0$. We will say that "the A-beam has been *phase-tuned* to the B-beam" when the composite beam is brightest, which we have just seen entails

$$\begin{aligned} \tan\phi &= \tan\theta_0 \\ &= i\,\frac{(E_B|E_A) - (E_A|E_B)}{(E_B|E_A) + (E_A|E_B)} \end{aligned} \tag{103}$$

To discover the more concrete meaning of this result, we retreat again to the $\longleftrightarrow\updownarrow$ representation, with respect to which

$$|E_A) \quad \text{acquires coordinates} \quad \begin{pmatrix} A_1 \\ A_2 e^{i\delta_A} \end{pmatrix}$$

$$|E_B) \quad \text{acquires coordinates} \quad \begin{pmatrix} B_1 \\ B_2 e^{i\delta_B} \end{pmatrix}$$

In this notation (103) becomes

$$\tan\phi = \frac{A_2 B_2 \sin(\delta_B - \delta_A)}{A_1 B_1 + A_2 B_2 \cos(\delta_B - \delta_A)} \tag{104}$$

which, we notice, remains unchanged if we make either beam brighter/dimmer: evidently the value of ϕ depends only upon the relative *figures* of the two beams;

i.e., upon their relative placement on the Poincaré sphere. In pursuit of this observation, let

$$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ describe the Poincaré vector of the } A\text{-beam}$$

$$\boldsymbol{b} \equiv \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ describe the Poincaré vector of the } B\text{-beam}$$

and notice that (104) can be written

$$\begin{aligned} \tan\phi &= \frac{4A_1A_2B_1B_2[\sin\delta_B\cos\delta_A - \cos\delta_B\sin\delta_A]}{[(A_1^2+A_2^2)+(A_1^2-A_2^2)][(B_1^2+B_2^2)+(B_1^2-B_2^2)]+4A_1A_2B_1B_2[\cos\delta_B\cos\delta_A+\sin\delta_B\sin\delta_A]} \\ &= \frac{a_2b_3 - a_3b_2}{(1+a_1)(1+b_1)+a_2b_2+a_3b_3} \qquad (105.1) \\ &= \frac{(\boldsymbol{a}\times\boldsymbol{b})_1}{1+(\boldsymbol{a}+\boldsymbol{b})_1+\boldsymbol{a}\cdot\boldsymbol{b}} \qquad (105.2) \end{aligned}$$

This result establishes that *Poincaré data (normalized Stokes data), though it contains no reference to temporal features of the underlying physical process, does convey enough information to permit description of the phase-tuning angle*.

It becomes useful at this point to adopt a sharpened notation: in place of ϕ write ϕ_{AB} and interpret the subscript to refer to the process "tune B so as to optimize (maximize the brightness of) $A\oplus B$." It is intuitively evident—and follows from (105)—that $\phi_{AB} = -\phi_{BA}$, of which

$$\phi_{AB} + \phi_{BA} = 0$$

provides a more symmetric formulation. Which brings me to the point of these remarks:

In 1956 S. Pancharatnam, working at the Raman Research Institute in Bangalore, was led from his experimental work to the theoretical observation[30] tht *beam-tuning is not a transitive procedure*; if B is tuned to A, and C to B, then it is generally *not* the case that C is tuned to A:

$$\phi_{ABC} \equiv \phi_{AB} + \phi_{BC} + \phi_{CA} \neq 0 \qquad (106)$$

A simple example serves to establish the point: take beams A, B and C whose Poincaré vectors are

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

[30] "Generalized theory of interference, and its applications," Proceedings of the Indian Academy of Sciences **44**, 247 (1956).

Working from (105) we then compute

$$\tan\phi_{AB} = \frac{0}{1+1+0} \quad \text{giving} \quad \phi_{AB} = 0$$
$$\tan\phi_{BC} = \frac{1}{1+0+0} \quad \text{giving} \quad \phi_{BC} = \tfrac{\pi}{4}$$
$$\tan\phi_{CA} = \frac{0}{1+1+0} \quad \text{giving} \quad \phi_{CA} = 0$$

So we have

$$\phi_{ABC} = \tfrac{\pi}{4} \neq 0$$

which establishes Pancharatnam's first point, and illustrates his second—deeper and more beautiful—discovery: the vectors $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$ mark a spherical triangle on Poincaré's unit sphere. That triangle is, in the present simple instance, a spherical octant, and has area $\Omega = \frac{1}{8}4\pi = \frac{\pi}{2}$. Pancharatnam observed it to be *generally* the case that

$$\begin{aligned}\phi_{ABC} &= \tfrac{1}{2}\Omega(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \\ \Omega(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) &\equiv \text{steradian } \textit{area} \text{ of the spherical triangle}\end{aligned} \tag{107}$$

Pancharatnam's theorem (107) attracted little notice[31] until after "Berry's phase" had entered the vocabulary of theoreticians,[32] and the theorem had been brought to the attention of Michael Berry by S. Ramaseshan & R. Nityananda.[33] It was in response to that news than Berry wrote the paper[34] which brought Pancharatnam's discovery to general attention, and made clear its relationship to subsequent developments.[35]

[31] Pancharatnam's paper is cited three times in Born & Wolf's *Principles of Optics* (6th edition 1980), but only to provide general support of the claim that the Stokes/Poincaré construction is sometimes useful; there is no evidence that either author has actually read the paper, or appreciated its significance.

[32] M. V. Berry, "Quantal phase factors accompanying adiabatic changes," Proc. Roy. Soc. (London) **A392**, 45 (1984). In "The quantum phase, five years after"—a retrospective essay published (together with reprints of most of the papers here cited) in A. Shapere & F. Wilczek, *Geometric Phases in Physics* (1989)—Berry remarks that it was a post-seminar question by R. Fox (Georgia Institute of Technology, spring 1983) which stimulated him to develop the mature "theory of Berry phase." Ronald Forrest Fox graduated in physics from Reed College in 1964, and was one of my own first students.

[33] Current Science, India **55**, 1225 (1986).

[34] "The adiabatic phase and Pancharatnam's phase for polarized light," J. Mod. Optics **34**, 1401 (1987).

[35] In the paper just cited, Berry argues that Pancharatnam's optical phase shift is "precisely analogous to the phase shift later predicted by Aharonov & Bohm" (1959), and is to be distinguished from the phase shift which results (R. Y. Chiao & Y. S. Wu, "Manifestations of Berry's topological phase for the photon," Phys. Rev. Lett. **57**, 933 (1986)) when the optical propagation vector $\boldsymbol{k}$ is made to trace a closed loop on the $\boldsymbol{k}$-sphere.

Pancharatnam himself was led to (107) by an argument of almost magical simplicity. He introduces a series of four elementary propositions relating in very physical terms to the superposition/resolution of beams. It is from the design of those propositions that he acquires his interest in triangles inscribed upon the Poincaré sphere. He then reaches into an obscure corner of an antique subject (spherical trigonometry)[36] to proceed from a formula in hand to "the following unexpected geometrical result." C. Brosseau[18] gives in his §3.2.4 an outline of what he calls "an elegant derivation of Pancharatnam's theorem due to Aravind"[37] which I have not seen, but which appears to be similarly rooted in spherical trigonometry. It is because *spherical triangles can have nothing ultimately to do with the phenomenon in question* that I prefer to proceed in the language of differential geometry.[38]

Let $\mathcal{C}$ be a closed curve inscribed on the surface S^2 of the unit 3-sphere, on which we have installed spherical coordinates $\{\varphi, \vartheta\}$ in the usual way.[39] The area Ω of the region $\mathcal{R}$ bounded by $\mathcal{C}$ can—subject to a certain proviso (see below)—be described

$$\Omega(\mathcal{R}) = \oint_{\mathcal{C}} \tfrac{1}{2}\big(\varphi \cos\vartheta \, d\vartheta - \sin\vartheta \, d\varphi\big) \tag{108}$$

This I assert on grounds that when $\mathcal{C}$ bounds the infinitesimal cell shown below

$$\begin{tikzcd}
(\varphi, \vartheta + d\vartheta) \arrow[d] & (\varphi + d\varphi, \vartheta + d\vartheta) \arrow[l] \\
(\varphi, \vartheta) \arrow[r] & (\varphi + d\varphi, \vartheta) \arrow[u]
\end{tikzcd}$$

the proposed formula yields

$$\begin{aligned}
\oint_{\mathcal{C}} \tfrac{1}{2}(\text{etc.}) &= \tfrac{1}{2}\Big\{\big[-\sin\vartheta \, d\varphi\big] + \big[(\varphi + d\varphi)\cos\vartheta \, d\vartheta\big] \\
&\qquad + \big[-(\sin\vartheta + \cos\vartheta \, d\vartheta)(-d\varphi)\big] + \big[\varphi\cos\vartheta(-d\vartheta)\big]\Big\} \\
&= \cos\vartheta \, d\varphi d\vartheta \\
&= \text{differential area } d\Omega \text{ of the spherical patch}
\end{aligned}$$

Notice, however, that if $\mathcal{C}$ is a circle of high latitude (i.e., if $\mathcal{R}$ caps the north pole) then (108) gives

$$\Omega(\text{polar cap}) = -\pi, \text{ even as cap radius} \downarrow 0$$

[36] W. J. M'Clelland & T. Preston, *A Treatise on Spherical Trigonometry with Applications to Spherical Geometry* (1897), Part II, Chapter 7, p. 50, Exercise 1.

[37] P. K. Aravind, Opt. Commun. **94**, 1992 (1992).

[38] As, according to Brosseau, was the preference also of E. De Vito & A. Lavrero in another paper (J. Mod. Opt. **41**, 233 (1994)) I have not seen.

[39] Which is to say: $x = \cos\varphi\cos\vartheta$, $y = \cos\varphi\sin\vartheta$, $z = \sin\vartheta$.

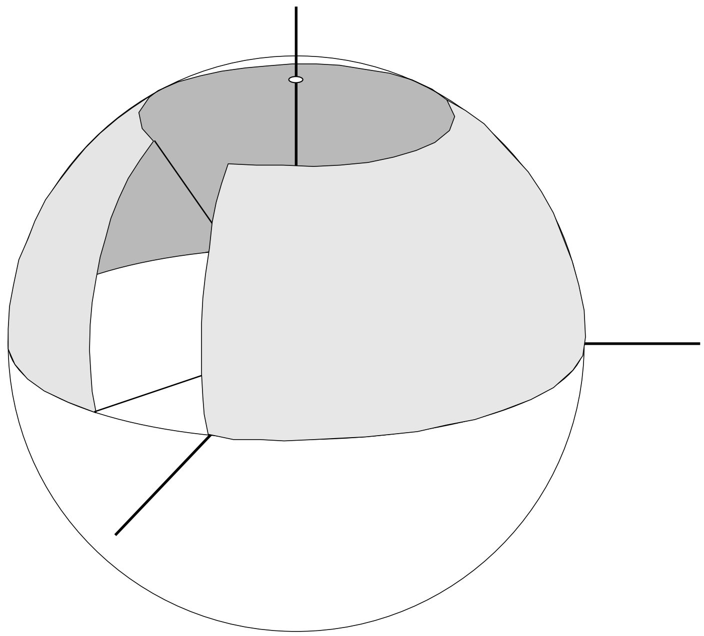

FIGURE 5: *Correct pole-avoidance procedure. The shaded region* $\mathcal{R}\left\{0 \leqslant \varphi \leqslant \hat{\varphi}\,;\, 0 \leqslant \vartheta \leqslant \hat{\vartheta} < \frac{\pi}{2}\right\}$ *has area*

$$\begin{aligned}\Omega &= \tfrac{1}{2}\Big\{\int_0^{\hat{\varphi}} 0\,d\varphi + \int_0^{\hat{\vartheta}} \hat{\varphi}\cos\vartheta\,d\vartheta + \int_{\hat{\varphi}}^0 (-\sin\hat{\vartheta})\,d\varphi + \int_{\hat{\vartheta}}^0 0\,d\vartheta\Big\}\\ &= \hat{\varphi}\sin\hat{\vartheta}\end{aligned}$$

Setting $\varphi = 2\pi$ *we therefore have*

$$\begin{aligned}\Omega(\textit{hemisphere}) &= \lim_{\hat{\vartheta}\to\frac{\pi}{2}} 2\pi\sin\hat{\vartheta} = 2\pi\\ \Omega(\textit{equatorial belt}) &= 2\pi\,d\vartheta\end{aligned}$$

Note particularly why it is—even in those extreme cases, and a casual reading of the figure notwithstanding—that the second and third integrals make identical contributions, and that the second and fourth fail to cancel.

which is absurd. And that if we use (108) to compute the area of the principal octant $\left\{0 \leqslant \varphi \leqslant \frac{\pi}{2}; 0 \leqslant \vartheta \leqslant \frac{\pi}{2}\right\}$ we are led to write

$$\begin{aligned}\Omega(\text{octant}) &= \tfrac{1}{2}\Big\{\int_0^{\frac{\pi}{2}} 0\,d\varphi + \tfrac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos\vartheta\,d\vartheta + \int_{\frac{\pi}{2}}^0 0\,d\vartheta\Big\}\\ &= \tfrac{\pi}{4}, \text{ which is only half the correct value}\end{aligned}$$

The moral is that (108) must be used *subject to the proviso that* $\mathcal{C}$ *does not envelop (first example) or pass through (second example) the polar singular points of the* $\{\varphi,\vartheta\}$ *coordinate system.* Correct procedures are illustrated in Figure 5.

Equation (108) refers to the spherical instance of a circumstance which is more familiar as encountered on the Euclidean plane: "Green's theorem"[40] reads

$$\iint_{\mathcal{R}} \big(A_{y,x} - A_{x,y}\big)\,dxdy = \oint_{\mathcal{C}} \big(A_x\,dx + A_y\,dy\big)$$

and in the special case $A_x = -\tfrac{1}{2}y$, $A_y = +\tfrac{1}{2}x$ gives back

$$\text{area of }\mathcal{R} = \tfrac{1}{2}\oint_{\mathcal{C}} \big(x\,dy - y\,dx\big) = \tfrac{1}{2}\oint_{\mathcal{C}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & y \\ 1 & x+dx & y+dy \end{vmatrix}$$

Note, however, that the expressions which appear on left and right in the statement of Green's theorem are, for superficially distinct reasons, invariant with respect to gauge transformations

$$\boldsymbol{A} \longrightarrow \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla} G \quad : \quad G(x,y) \text{ arbitrary}$$

and that in

$$\text{area of }\mathcal{R} = \tfrac{1}{2}\oint_{\mathcal{C}} \Big\{\big(-y + \tfrac{\partial}{\partial x}G\big)\,dx + \big(x + \tfrac{\partial}{\partial y}G\big)\,dy\Big\}$$

we have a *multitude* of seemingly distinct descriptions of plane area. Nor is this development special to the Euclidean case; in place of (108) we have the more general statement[41]

$$\Omega(\text{spherical }\mathcal{R}) = \oint_{\mathcal{C}} \Big\{\big(A_1(\varphi,\vartheta) + \tfrac{\partial}{\partial\varphi}G\big)\,d\varphi + \big(A_2(\varphi,\vartheta) + \tfrac{\partial}{\partial\vartheta}G\big)\,d\vartheta\Big\} \tag{109.1}$$

where

$$\left.\begin{aligned} A_1(\varphi,\vartheta) &\equiv -\tfrac{1}{2}\sin\vartheta \\ A_2(\varphi,\vartheta) &\equiv \tfrac{1}{2}\,\varphi\cos\vartheta \end{aligned}\right\} \tag{109.2}$$

and the gauge function $G(\varphi,\vartheta)$ is arbitrary.

Reverting now to (105), if

$$\boldsymbol{a} = \begin{pmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\,\sin\varphi \\ \sin\vartheta \end{pmatrix}$$

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[40] See CLASSICAL ELECTRODYNAMICS (1980), p. 78.

[41] One might expect to find such a result in R. Creighton Buck's wonderful *Advanced Calculus* (1956), though I have been unable to.

and $\boldsymbol{b}$ lies close by

$$\boldsymbol{b} = \boldsymbol{a} + \delta\boldsymbol{a} \quad \text{with} \quad \delta\boldsymbol{a} = \begin{pmatrix} -\sin\vartheta\cos\varphi\, d\vartheta - \cos\vartheta\,\sin\varphi\, d\varphi \\ -\sin\vartheta\,\sin\varphi\, d\vartheta + \cos\vartheta\cos\varphi\, d\varphi \\ \cos\vartheta\, d\vartheta \end{pmatrix}$$

then

$$\begin{aligned} d\phi &\equiv \text{Pancharatnam's phase differential} \\ &= \frac{\sin\varphi\, d\vartheta - \sin\vartheta\cos\vartheta\cos\varphi\, d\varphi}{2(1+\cos\vartheta\cos\varphi)} \\ &\equiv B_1(\varphi,\vartheta)\, d\varphi + B_2(\varphi,\vartheta)\, d\vartheta \end{aligned} \tag{110}$$

What, from this point of view, Pancharatnam discovered (and direct calculation readily confirms) is that

$$\boldsymbol{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{and} \quad 2\boldsymbol{B} = 2\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{are gauge-equivalent:}$$

$$\tfrac{\partial}{\partial\vartheta}\big(A_1 - 2B_1\big) - \tfrac{\partial}{\partial\varphi}\big(A_2 - 2B_2\big) = 0 \tag{111}$$

All the mensuration formulæ of spherical trigonometry are implicit in (110), but the argument just concluded proceeds with out reference to such extraneous details; as a generalization of (107) we have

$$\phi(\text{loop}) = \oint_{\mathcal{C}} d\phi = \tfrac{1}{2}(\text{steradian loop area}) \tag{112}$$

We are in position now to appreciate Berry's train of thought as developed in "§3. Aharonov-Bohm effect on the Poincaré sphere" of a paper cited earlier.[34] Let a monochromatic beam in the state described by the Poincaré vector $\boldsymbol{a}$ be presented to a beam-splitter. One emergent beam proceeds unimpeded. The other is tickled through a λ-parameterized sequence of states

$$\boldsymbol{a}(\lambda) \quad : \quad \boldsymbol{a} = \boldsymbol{a}(0) \looparrowright \boldsymbol{a}(1) = \boldsymbol{a} \text{ again}$$

and then reunited with its companion beam. What Pancharatnam has in effect demonstrated is that the reunited beams will be out of phase,[42] and that

$$\text{phase difference} = \tfrac{1}{2}\big(\text{area of region patrolled by } \boldsymbol{a}(\lambda)\big)$$

Berry draws attention to the elementary fact that if we deposit an "abstract monopole of strength $-\frac{1}{2}$ at the center of the Poincaré sphere" then area becomes interpretable as flux, and we will have reproduced the essentials of the Aharanov-Bohm effect.

[42] From (100) and (112) we see more particularly that the adventuresome beam will be relatively *retarded/advanced* according as the loop area is positive $\circlearrowleft$ or negative $\circlearrowright$.

9. Stokes' parameters in statistical optics. We saw in §5 that physical features characteristic of quasi-monochromatic beams lend a slow temporal wander to Stokes' parameters, and that the time-averaged parameters $\langle S_\mu \rangle$ supply useful information about the statistical properties of the beam; in particular, they permit one to construct a theory of *partial polarization*. In the following discussion we relax the quasi-monochromaticity assumption which gave rise to "slow temporal wander" and find that Stokes parameters nevertheless emerge as useful natural constructs, that "partial polarization" becomes linked to a concept of "partial coherence." The latter notion springs from the statistical theory of "signals" (time series)—a sprawling Amazon of a subject which flows through a jungle first explored by Norbert Wiener and cohorts in the 1930's. Even the tributary specific to optics (fed by laser technology and the practical needs of optical and radio astronomers) is of awesome scale. I propose to "walk on the rocks," venturing just far enough from shore to acquire one specific result; most details, essential qualifications and fine distinctions I must be content to leave to the literature.[43]

Let $f(t)$ be a real-valued signal. It is our cultivated instinct to write

$$f(t) = \text{real part of } \phi(t) \equiv f(t) + ig(t) \tag{113}$$

but before we can yield to such an impulse we must be in position to ascribe meaning/value to the function $g(t)$. One way to get into such a position is to proceed as follows: write

$$f(t) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega t}\, d\omega$$

The reality of $f(t)$ implies $F(-\omega) = F^*(\omega)$, so negative frequency data is redundant with positive requency data. We therefore expect the *complex* signal

$$\phi(t) \equiv \underbrace{\tfrac{2}{\sqrt{2\pi}} \int_0^{\infty} F(\omega) e^{-i\omega t}\, d\omega}_{\text{defines the so-called ``analytic signal''}} \tag{114}$$

to convey information identical to that written onto $f(t)$. To do so is, in effect, to write (113) with

$$g(t) = \tfrac{1}{\pi} P \int_{-\infty}^{+\infty} \tfrac{1}{s-t} f(s)\, ds \quad : \quad \text{``Hilbert transform'' of } f(t) \tag{115}$$

---

[43] Some of the missing detail, and essential references, can be found in OPTICS (1982), pp. 116–139.

If, in particular,

$$f(t) = \cos\omega t = +\tfrac{1}{2}e^{+i\omega t} + \tfrac{1}{2}e^{-i\omega t}$$

then

$$\phi(t) = e^{-i\omega t}$$

and[44]

$$ig(t) = -\tfrac{1}{2}e^{+i\omega t} + \tfrac{1}{2}e^{-i\omega t} = -i\sin\omega t$$

One can establish without much difficulty that

$$\int_{-\infty}^{+\infty} f^2(t)\,dt = \int_{-\infty}^{+\infty} g^2(t)\,dt = \tfrac{1}{2}\int_{-\infty}^{+\infty} |\phi(t)|^2\,dt$$

and (with more difficulty) why it is that in typical applications $\int f^2\,dt$ has something to do with "intensity." The moral is that importation of the "analytic signal" purchases analytical advantages, but never leads one far astray from the physics of the matter. Information concerning the coherence properties of a signal is conveyed by expressions of the form

$$\int \phi(t+\tau_1)\cdots\phi(t+\tau_\mu)\phi^*(t+\tau_{\mu+1})\cdots\phi^*(t+\tau_{\mu+\nu})\,dt$$

An optical beam can be construed to be a *bundled pair* of real signals $E_1(t)$ and $E_2(t)$, representable as a bundled pair of analytic signals $\phi_1(t)$ and $\phi_2(t)$. Interest in the dominant (leading order) coherence properties of such a beam leads to study of expressions of the type

$$c_{mn}(\tau, t) \equiv \lim_{T\to\infty} \tfrac{1}{T}\int_{t-T}^{t} E_m(t)E_n(t+\tau)\,dt$$

where the t-dependence drops away for the "steady" (or statistically stationary) beams in which we will have special interest, leaving

$$c_{mn}(\tau) = \langle E_m(t)E_n(t+\tau)\rangle$$

We are, in light of previous remarks, not surprised to discover that it proves to be analytically more convenient (but ultimately equivalent) to examine the elements

$$\mathit{\Gamma}_{mn}(\tau) \equiv \langle\phi_m(t)\phi_n^*(t+\tau)\rangle \tag{116.1}$$

of the so-called *correlation matrix*

$$\boldsymbol{\Gamma}(\tau) = \begin{pmatrix} \mathit{\Gamma}_{11}(\tau) & \mathit{\Gamma}_{12}(\tau) \\ \mathit{\Gamma}_{21}(\tau) & \mathit{\Gamma}_{22}(\tau) \end{pmatrix} \tag{116.2}$$

which are susceptible to observation by a fairly cunningly designed array of

[44] See **15.2.47** in A. Erdélyi *et al*, *Tables of Integral Transforms* (1954).

interferometric and photometric techniques;[45] the diagonal elements provide leading measures of the degree to which the respective component beams are "auto-correlated" (or temporally coherent), while the off-diagonal elements quantify the "cross-correlation." The correlation matrix evidently satisfies the "cross-hermiticity condition" $\boldsymbol{\Gamma}^\dagger(\tau) = \boldsymbol{\Gamma}(-\tau)$, but becomes hermitian in the standard sense at $\tau = 0$. Because to do so is to be led most directly to my intended objective, I restrict my attention henceforth to that special case, writing

$$\boldsymbol{\Gamma} \equiv \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \equiv \boldsymbol{\Gamma}(0) \tag{117}$$

The diagonal elements $\Gamma_{mm} = \langle \phi_m(t)\phi_m^*(t)\rangle$ are real, and refer to the mean *intensities* I_m of the component beams, while the off-diagonal elements are complex conjugates of one another, and refer as before to inter-component *correlation*. An informative benchmark is provided by the monochromatic signal (2), in which case we have

$$\boldsymbol{\Gamma} = 2\begin{pmatrix} \mathcal{E}_1\mathcal{E}_1 & \mathcal{E}_1\mathcal{E}_2 e^{-i\delta} \\ \mathcal{E}_2\mathcal{E}_1 e^{+i\delta} & \mathcal{E}_2\mathcal{E}_2 \end{pmatrix} \tag{118}$$

From

$$\langle(\xi\phi_1 + \phi_2)(\xi\phi_1 + \phi_2)^*\rangle = \xi^2\Gamma_{11} + \xi(\Gamma_{12} + \Gamma_{21}) + \Gamma_{22} \geqslant 0 \quad : \quad \text{all } \xi$$

we have

$$(\Gamma_{12} + \Gamma_{21})^2 \leqslant 4\Gamma_{11}\Gamma_{22}$$

which if we write $\Gamma_{12} = \sqrt{\Gamma_{11}\Gamma_{22}}\,\gamma e^{-i\theta} = (\Gamma_{21})^*$ becomes

$$(\gamma\cos\theta)^2 \leqslant 1 \tag{119.1}$$

$$\gamma \equiv \sqrt{\frac{\Gamma_{12}\Gamma_{21}}{\Gamma_{11}\Gamma_{22}}} \equiv \text{ dimensionless "degree of cross-correlation"} \tag{119.2}$$

In this notation

$$\begin{aligned} I = \langle(\phi_1 + \phi_2)(\phi_1 + \phi_2)^*\rangle &= \Gamma_{11} + \Gamma_{22} + \Gamma_{12} + \Gamma_{21} \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2}\cdot\gamma\cos\theta \end{aligned} \tag{120}$$

[45] Note, however, that

$$\begin{pmatrix} E_1(t) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ E_2(t) \end{pmatrix} \quad \text{do not interfere!}$$

To get anywhere interferometrically one must first of all contrive to achieve (say)

$$\begin{pmatrix} 0 \\ E_2(t) \end{pmatrix} \longrightarrow \begin{pmatrix} E_2(t) \\ 0 \end{pmatrix}$$

The parameter γ acquires its name from the observation that

- if $\gamma = 0$ then (typically of uncorrelated signals) intensities add (interference effects are absent), while
- if $\gamma = 1$ then interference effects are fully developed.

From (120) it follows that

$$\begin{aligned} I_{\max} &= I_1 + I_2 + 2\sqrt{I_1 I_2}\,\gamma \\ I_{\min} &= I_1 + I_2 - 2\sqrt{I_1 I_2}\,\gamma \end{aligned}$$

so in the language of observational interferometry we have

$$\text{“visibility”} \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{\sqrt{I_1 I_2}}{\frac{1}{2}(I_1 + I_2)}\,\gamma$$

It is, however, a general proposition[46] that *the arithmetic mean dominates the harmonic mean*: if a and b are any positive numbers, then

$$\frac{\sqrt{ab}}{\frac{1}{2}(a+b)} \leqslant 1, \text{ with equality if and only if } a = b$$

So we have

$$\text{visibility} \leqslant \gamma, \text{ with equality if and only if } I_1 = I_2$$

which suggests a method for *measuring* the degree γ of cross-correlation present in the beam. What about θ?

Because $\boldsymbol{\Gamma}$ is hermitian we can write $\boldsymbol{\Gamma} = \frac{1}{2}\big(\Gamma_0\boldsymbol{\sigma}_0 + \Gamma_1\boldsymbol{\sigma}_1 + \Gamma_2\boldsymbol{\sigma}_2 + \Gamma_3\boldsymbol{\sigma}_3\big)$, which in Pauli's representation becomes

$$\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} \Gamma_0 + \ \Gamma_1 & \Gamma_2 - i\Gamma_3 \\ \Gamma_2 + i\Gamma_3 & \Gamma_0 - \ \Gamma_1 \end{pmatrix} \tag{121}$$

giving

$$\left.\begin{aligned} \Gamma_0 &= \tfrac{1}{2}\big(\Gamma_{11} + \Gamma_{22}\big) & &\longrightarrow \ \ \mathcal{E}_1^2 + \mathcal{E}_2^2 \\ \Gamma_1 &= \tfrac{1}{2}\big(\Gamma_{11} - \Gamma_{22}\big) & &\longrightarrow \ \ \mathcal{E}_1^2 - \mathcal{E}_2^2 \\ \Gamma_2 &= \tfrac{1}{2}\big(\Gamma_{21} + \Gamma_{12}\big) = \ \ \sqrt{\Gamma_{11}\Gamma_{22}}\,\gamma\cos\theta & &\longrightarrow \ \ 2\mathcal{E}_1\mathcal{E}_2\cos\delta \\ i\Gamma_3 &= \tfrac{1}{2}\big(\Gamma_{21} - \Gamma_{12}\big) = i\,\sqrt{\Gamma_{11}\Gamma_{22}}\,\gamma\sin\theta & &\longrightarrow i\,2\mathcal{E}_1\mathcal{E}_2\sin\delta \end{aligned}\right\} \tag{122}$$

where I have used arrows $\longrightarrow$ to indicate what happens when one looks in particular to the monochromatic case (118). “What happens” is that one

---

[46] See p. 362 in R. Courant & H. Robbins' *What is Mathematics?* (1941), where the point is phrased this way: the rectangle of given perimeter and maximal area is square: $ab \leqslant \big[\frac{1}{2}(a+b)\big]^2$, with equality if and only if $a = b$.

recovers the expressions which at (20) served to define the Stokes parameters of a monochromatic beam! Evidently the "Pauli coordinates" of $\boldsymbol{\Gamma}$ serve in effect to generalize those definitions. The relative phase parameter δ (which itself presumes monochromaticity) has, at the same time, acquired a generalized meaning θ, but a meaning which still admits of polarimetric quantification. We are led by these remarks to make a notational adjustment $\Gamma_\mu \mapsto S_\mu$, and in place of (121) to write

$$\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} S_0 + \ S_1 & S_2 - iS_3 \\ S_2 + iS_3 & S_0 - \ S_1 \end{pmatrix} \tag{123}$$

From

$$\begin{aligned} \det \boldsymbol{\Gamma} = \Gamma_{11}\Gamma_{22}\big\{1-\gamma^2\big\} &= S_0^2 - S_1^2 - S_2^2 - S_3^2 \\ &= S_0^2\big\{1 - P^2\big\} \\ &= \big[\tfrac{1}{2}(\Gamma_{11}+\Gamma_{22})\big]^2\big\{1-P^2\big\} \\ &\geqslant \Gamma_{11}\Gamma_{22}\big\{1-P^2\big\} \geqslant 0 \end{aligned}$$

we obtain $1 \geqslant 1-\gamma^2 \geqslant 1 - P^2 \geqslant 0$ giving

$$0 \leqslant \gamma \leqslant P \leqslant 1 \tag{124}$$

The central inequality asserts that the

$$\text{degree of cross-correlation} \leqslant \text{degree of polarization}$$

with equality if and only if $\Gamma_{11} = \Gamma_{22}$, which at (122) was seen to entail $S_1 = 0$. For an uncorrelated beam $\gamma = 0$, which by (122) entails $S_2 = S_3 = 0$, while for an unpolarized beam $S_1 = S_2 = S_3 = 0$.

We are brought thus to the conclusion that

$$\begin{aligned} P^2 &= \frac{S_1^2 + S_2^2 + S_3^2}{S_0^2} \\ \gamma^2 &= 1 - \frac{S_0^2}{S_0^2 - S_1^2}\big(1 - P^2\big) = \frac{S_2^2 + S_3^2}{S_0^2 - S_1^2} \\ \theta &= \arctan\frac{S_3}{S_2} \end{aligned}$$

refer in their separate ways to information borne by Stokes' parameters S_μ, and that the latter, in their most general guise, refer to leading-order auto/cross-correlation properties of the 2-component beam.

Given an n-component signal we expect to be led by similar argument to an expanded set of parameters

$$\big\{S_0, S_1, \ldots, S_{n^2-1}\big\}$$

Remarks pertaining to the case $n = 3$ can be found in §3.1.6.7 and Appendix D of Brosseau.[18]

Beams with identical Stokes parameters (identical leading-order statistics) can be expected generally to be distinct in higher order. Fourth-order statistics bears directly upon some important physics (theory of intensity fluctuations, speckle interferometry, the Brown-Twiss effect), but the mathematics becomes rapidly more complicated as the statistical order ascends, partly because simple matrix methods no longer suffice to bear the analytical burden.

I have indicated how the Mueller calculus (§4) and the Jones calculus (§6) can be used

- to describe the state of a monochromatic beam
- to describe the action upon such a beam by a linear device

and have written "Jones = $\sqrt{\text{Mueller}}$" to suggest the relationship between those calculi. We are in position now to speak more precisely about the features which distinguish those formalisms:

The objects fundamental to the Mueller calculus are Stokes' parameters, which are quadratic in the fields and directly observable as intensities. Within that formalism one writes

$$S_\mu^{\text{superimposed beams}} = \sum S_\mu^{\text{component beam}}$$

but the presumption that "intensities add" was seen at (120) to entail a presumption that the component beams are uncorrelated: $\gamma = 0$. The theory initially presumed monochromaticity, but was found by introduction of "slow temporal wander" to support a notion of partial polarization.

The objects fundamental to the Jones calculus, on the other hand, are the fields themselves (amplitudes and relative phase, with a shared $e^{i\omega t}$-factor discarded), and to describe beam superposition one writes

$$|E)^{\text{superimposed beams}} = \sum |E)^{\text{component beam}}$$

A monochromaticity assumption is central to the formalism, and since

- monochromaticity $\Longrightarrow$ perfect correlation: $\gamma = 1$
- $\gamma = 1 \Longrightarrow P = 1$, by (124)

the Jones calculus is rendered inapplicable to partially polarized beams. But the theory extracted from the correlation matrix $\boldsymbol{\Gamma}$, written on complex 2-space as it is, does possess a very Jonesesque coloration, and can be considered to have remedied that defect.

10. Beam entropy. From the $\boldsymbol{\Gamma} = S_0\boldsymbol{\sigma}_0 + S_1\boldsymbol{\sigma}_1 + S_2\boldsymbol{\sigma}_2 + S_3\boldsymbol{\sigma}_3$ characteristic of some given beam, form the "density matrix"

$$\boldsymbol{\rho} \equiv \tfrac{1}{2S_0}\boldsymbol{\Gamma} = \tfrac{1}{2}\big(\mathbf{I} + s_1\boldsymbol{\sigma}_1 + s_2\boldsymbol{\sigma}_2 + s_3\boldsymbol{\sigma}_3\big) = \tfrac{1}{2}\begin{pmatrix} 1 + s_1 & s_2 - is_3 \\ s_2 + is_3 & 1 - s_1 \end{pmatrix} \tag{125}$$

which, by contrivance, has unit trace. The associated beam has polarization

$$P = \sqrt{s_1^2 + s_2^2 + s_3^2} \leqslant 1 \tag{126}$$

and (to say the same thing another way) the 3-vector $\boldsymbol{s}$ marks a point
- on the surface of the Poincaré sphere if $P = 1$ (perfect polarization)
- interior to the Poincaré sphere if $P < 1$ (partial polarization).

Quite generally

$$\det(\mathsf{M} - \lambda\mathsf{I}) = \lambda^2 - (\operatorname{tr}\mathsf{M})\lambda + \det\mathsf{M} \qquad \text{if } \mathsf{M} \text{ is } 2\times 2$$

so in the case at hand

$$\det(\boldsymbol{\rho} - \lambda\mathsf{I}) = \lambda^2 - \lambda + \tfrac{1}{4}(1 - P^2) \tag{127}$$

The eigenvalues of $\boldsymbol{\rho}$ are given therefore by

$$\lambda = \tfrac{1}{2}(1 \pm P) \quad \text{which} \begin{cases} \text{become } \{0, 1\} \text{ in the case } P = 1 \\ \text{become } \{\tfrac{1}{2}, \tfrac{1}{2}\} \text{ in the case } P = 0 \\ \text{sum to unity in all cases} \end{cases} \tag{128}$$

It follows from (127) by the Cayley-Hamilton theorem that

$$\boldsymbol{\rho}^2 = \boldsymbol{\rho} - \tfrac{1}{4}(1 - P^2)\,\mathsf{I}$$

according to which $\boldsymbol{\rho}$ becomes projective in the case $P = 1$. If $P \neq 0$ we can in contrary cases form

$$\mathsf{P} \equiv \tfrac{1}{2}\big(\mathsf{I} + \hat{s}_1\boldsymbol{\sigma}_1 + \hat{s}_2\boldsymbol{\sigma}_2 + \hat{s}_3\boldsymbol{\sigma}_3\big) \quad : \quad \hat{\boldsymbol{s}} \equiv \tfrac{1}{P}\boldsymbol{s} \text{ is a unit vector} \tag{129.1}$$

and observe that, for the reasons just described, P is projective. So also therefore is

$$\mathsf{P}_\perp \equiv \mathsf{I} - \mathsf{P} = \tfrac{1}{2}\big(\mathsf{I} - \hat{s}_1\boldsymbol{\sigma}_1 - \hat{s}_2\boldsymbol{\sigma}_2 - \hat{s}_3\boldsymbol{\sigma}_3\big) \tag{129.2}$$

which (trivially) is orthogonal to P: $\mathsf{P}\mathsf{P}_\perp = \mathsf{O}$. We are in position now to notice that

$$\begin{aligned} &\tfrac{1}{2}(1+P)\,\mathsf{P} + \tfrac{1}{2}(1-P)\,\mathsf{P}_\perp \\ &\qquad = \tfrac{1}{2}\big[\tfrac{1}{2}(1+P) + \tfrac{1}{2}(1-P)\big]\mathsf{I} + \tfrac{1}{2}\big[\tfrac{1}{2}(1+P) - \tfrac{1}{2}(1-P)\big]\hat{\boldsymbol{s}}\boldsymbol{\cdot}\boldsymbol{\sigma} \\ &\qquad = \tfrac{1}{2}\mathsf{I} + \tfrac{1}{2}P\,\hat{\boldsymbol{s}}\boldsymbol{\cdot}\boldsymbol{\sigma} \\ &\qquad = \tfrac{1}{2}\big(\mathsf{I} + \boldsymbol{s}\boldsymbol{\cdot}\boldsymbol{\sigma}\big) \\ &\qquad = \boldsymbol{\rho} \end{aligned} \tag{130}$$

at which point we have accomplished the *spectral resolution* of $\boldsymbol{\rho}$: we have displayed the density matrix as a weighted sum[47]

$$\boldsymbol{\rho} = p_1\mathsf{P}_1 + p_2\mathsf{P}_2 \tag{131}$$

[47] Here—the better to expose the simple point—I have adopted a simplified notation intended to eliminate some distracting clutter.

of orthogonal projection opertors, and the associated partially polarized beam as an incoherent superposition of a pair of oppositely polarized beams.

It becomes at this point entirely natural to mimic the procedure by which John von Neumann assigned an "entropy" to quantum mechanical mixed states; i.e., to introduce a

$$\text{“beam entropy”} \equiv -p_1 \log p_1 - p_2 \log p_2 \tag{132}$$

$$\begin{aligned} &= -\tfrac{1}{2}(1+P)\log\left[\tfrac{1}{2}(1+P)\right] - \tfrac{1}{2}(1-P)\log\left[\tfrac{1}{2}(1-P)\right] \\ &= -\log\left\{\left[\tfrac{1}{2}(1+P)\right]^{\frac{1}{2}(1+P)}\left[\tfrac{1}{2}(1-P)\right]^{\frac{1}{2}(1-P)}\right\} \qquad (133) \\ &= \begin{cases} 0 & \text{at } P=1\text{: beam “maximally organized”} \\ \log 2 & \text{at } P=0\text{: beam “minimally organized”} \end{cases} \end{aligned}$$

Evidently "degree of polarization" and "beam entropy" speak in distinct but equivalent ways about the same thing. The right side of (133) is plotted in the following figure.

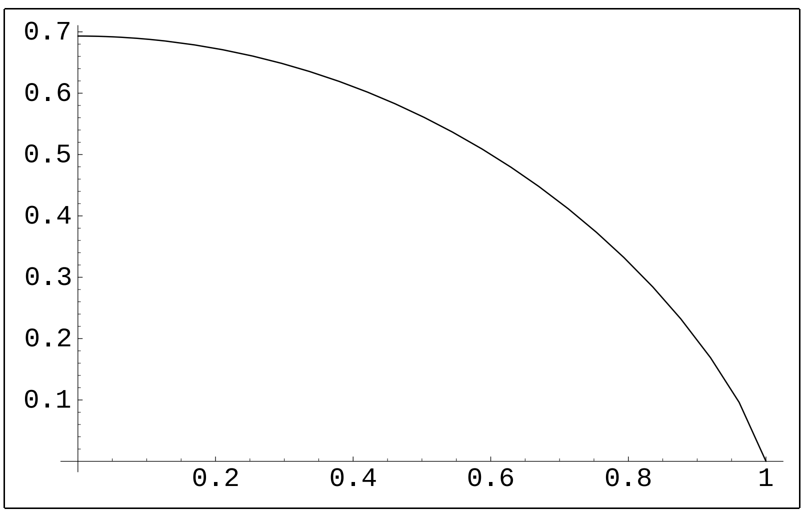

FIGURE 6: *"Beam entropy" plotted vs."degree of polarization."*

I describe now an alternative organization of the derivation of (132). We take as our points of departure the observations that (132) can be written

$$\text{“beam entropy”} = -\mathrm{tr}\big\{\boldsymbol{\rho}\log\boldsymbol{\rho}\big\} \tag{134}$$

and that (130) can be expressed[48]

$$\begin{aligned} \boldsymbol{\rho} &= P\,\mathbf{P} + \tfrac{1}{2}(1-P)(\mathbf{P}+\mathbf{P}_\perp) \\ &= \tfrac{1}{2}(1-P)\,\mathbf{I} + P\,\mathbf{P} \qquad (135) \\ &= \boldsymbol{\rho}_{\text{unpolarized}} + \boldsymbol{\rho}_{100\%\text{ polarized}} \end{aligned}$$

[48] See again the remark immediately subsequent to (51).

How to describe $\log\boldsymbol{\rho}$? Notice that if P is projective then

$$\begin{aligned} e^{a\mathsf{I}+b\mathsf{P}} = e^{a\mathsf{I}}\cdot e^{b\mathsf{P}} &= e^a\big\{\mathsf{I}+(e^b-1)\mathsf{P}\big\} \\ &= \alpha\,\mathsf{I}+\beta\,\mathsf{P} \quad\text{with}\quad \alpha = e^a,\ \beta = e^a(e^b-1) \end{aligned}$$

so

$$\log(\alpha\,\mathsf{I}+\beta\,\mathsf{P}) = (\log\alpha)\cdot\mathsf{I} + \log(1+\beta/\alpha)\cdot\mathsf{P} \tag{136}$$

From this general proposition it follows in particular[49] that

$$\log\boldsymbol{\rho} = \big(\log\tfrac{1-P}{2}\big)\cdot\mathsf{I} + \big(\log\tfrac{1+P}{1-P}\big)\cdot\mathsf{P}$$

giving

$$\begin{aligned} \boldsymbol{\rho}\log\boldsymbol{\rho} &= \Big\{\big(\tfrac{1-P}{2}\big)\cdot\mathsf{I}+\big(P\big)\cdot\mathsf{P}\Big\}\Big\{\big(\log\tfrac{1-P}{2}\big)\cdot\mathsf{I}+\big(\log\tfrac{1+P}{1-P}\big)\cdot\mathsf{P}\Big\} \\ &= \big(\tfrac{1-P}{2}\log\tfrac{1-P}{2}\big)\cdot\mathsf{I} + \Big\{\tfrac{1-P}{2}\log\tfrac{1+P}{1-P} + P\big[\log\tfrac{1-P}{2}+\log\tfrac{1+P}{1-P}\big]\Big\}\cdot\mathsf{P} \\ &= \big(\tfrac{1-P}{2}\log\tfrac{1-P}{2}\big)\cdot\mathsf{I} + \Big\{\tfrac{1+P}{2}\log\tfrac{1+P}{2} - \tfrac{1-P}{2}\log\tfrac{1-P}{2}\Big\}\cdot\mathsf{P} \end{aligned}$$

But (because I is 2×2) $\mathrm{tr}\,\mathsf{I} = 2$ and (because P projects onto a ray) $\mathrm{tr}\,\mathsf{P} = 1$, so

$$-\mathrm{tr}\big\{\boldsymbol{\rho}\log\boldsymbol{\rho}\big\} = \text{expression displayed at (133)}$$

This computational strategy pertains also to (131) as it pertains, indeed, to any representation of the design $\boldsymbol{\rho} = \sum$(commuting projectors).

When the possibility of defining a "beam entropy" first occurred to me I naively thought the idea to be novel. I discover, however, that it is an idea with a history, and that it is especially dear to Christian Brosseau, who made it the subject of Chapter 3.4 in his recent monograph,[18] and returns to the topic in his §§4.1.1.9, 4.1.2.6 and Appendix H. Though Brosseau does make some use of the idea in connection with his account of the theory of beam degradation by multiple scattering, it is my impression that "beam entropy" remains at the moment a pretty idea with nothing to do, waiting to be folded into some future theory: it serves as a beam descriptor (equivalent to P), but we are

- in possession of no statement of the form

$$\text{beam entropy} = \log(\text{number of equivalent states})$$

- in no position to speak of "entropy *transport*"
- in no position to write out the thermodynamics of *thermalized* beams.

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[49] Reading from (135) we set $\alpha = \frac{1}{2}(1-P)$ and $\beta = P$, which entail

$$1+\beta/\alpha = \frac{1+P}{1-P}$$

We recall that it was the thermodynamics of thermalized (blackbody) radiation that gave the world quantum mechanics. But "blackbody beams" are maximally featureless; they are, in particular, unpolarized, so the relationship (if any) between Planck's "modal entropy density" and "beam entropy" is necessarily indirect.

Some algebraic aspects of the preceding discussion will have filled readers with a sense of *déjà vu*. I discuss now why that is so. In my account (§6) of the Jones calculus I had occasion to introduce (at (73)) what I idiosyncratically called the "beam operator"

$$\mathbf{b} \equiv \tfrac{1}{2}\big\{S_0\boldsymbol{\sigma}_0 + S_1\boldsymbol{\sigma}_1 + S_2\boldsymbol{\sigma}_2 + S_3\boldsymbol{\sigma}_3\big\}$$

Operative at the time was an explicit monochromaticity assumption (which entailed 100% polarization, whence $S_0^2 - S_1^2 - S_2^2 - S_3^2$), but that assumption can be dropped; one arrives then at a "beam discriptor" which is proportional to the "density operator" $\boldsymbol{\rho}$:

$$\mathbf{b} = S_0 \cdot \underbrace{\tfrac{1}{2}\big\{\mathbf{I} + s_1\boldsymbol{\sigma}_1 + s_2\boldsymbol{\sigma}_2 + s_3\boldsymbol{\sigma}_3\big\}}_{\boldsymbol{\rho}}$$

The question arises: Why, in optical theory, does one need *both* $\mathbf{b}$ and $\boldsymbol{\rho}$? The density operator $\boldsymbol{\rho}$ is the "mathematically more natural" of the two (it has unit trace, and becomes projective in the "pure case"), and it is the object familiar from quantum theory. But in quantum theory the value of $(\psi|\psi)$ is universal, while in optics the value of $(E|E)$ is variable. To distinguish bright beams from dim beams of otherwise identical design one needs, in addition to the information conveyed by $\boldsymbol{\rho}$, the absolute intensity information conveyed by S_0, and $\mathbf{b}$ is simply the "name" of the duplex construct $S_0\mathbf{b}$.

Aspects of the preceding discussion—particularly the relation of (135) to (131)—inspire the following mathematical digression:

11. Principle of optical equivalence, revisited. I begin with remarks bearing upon my recent claim (end of §9) that "the Jones calculus is... inapplicable to partially polarized beams." Relative to the $\{\longleftrightarrow\updownarrow\}$-basis the Jones vector $|E)$ representative of a monochromatic (!) beam acquires the coordinates

$$\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2 e^{i\delta}\end{pmatrix}$$

Multiplication by $\mathcal{E}_1$ and appeal the the equations (20) which served initially to introduce Stokes' parameters gives

$$\begin{aligned}\mathcal{E}_1\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2 e^{i\delta}\end{pmatrix} &= \tfrac{1}{2}\begin{pmatrix}S_0 + \ S_1\\ S_2 + iS_3\end{pmatrix} \quad \text{with} \quad S_0^2 = S_1^2 + S_2^2 + S_3^2\\ &= \tfrac{1}{2}S_0\begin{pmatrix}1 \ + \ \hat{s}_1\\ \hat{s}_2 + i\hat{s}_3\end{pmatrix}\end{aligned}$$

We have, therefore, a nest of associations

$$\begin{array}{c}\text{monochromatic beam}\\ \downarrow\\ \text{point } \hat{\boldsymbol{s}} \text{ on } \textit{surface} \text{ of the Poincaré sphere}\end{array}$$

$$\mathsf{P} = \tfrac{1}{2}\begin{pmatrix} 1 + \hat{s}_1 & \hat{s}_2 - i\hat{s}_3 \\ \hat{s}_2 + i\hat{s}_3 & 1 - \hat{s}_1 \end{pmatrix} \qquad\qquad \xi = \begin{pmatrix} 1 + \hat{s}_1 \\ \hat{s}_2 + i\hat{s}_3 \end{pmatrix}$$

which is brought to closure by the not-very-surprising observation that

$$\mathsf{P}\xi = \xi \quad : \quad \mathsf{P} \text{ projects onto the } \xi\text{-ray}$$

The transformation $\hat{\boldsymbol{s}} \to -\hat{\boldsymbol{s}}$ (which sends a point on the Poincaré sphere to its diametric opposite, and which at (129) was seen to send $\mathsf{P} \to \mathsf{P}_\perp$) sends

$$\xi \to \xi_\perp \quad : \quad \mathsf{P}\xi_\perp = 0 \;\text{ and }\; \xi_\perp^*\xi = 0$$

All of which—physics and formalism—lives on the 100% polarized *surface* of the Poincaré sphere, and in connection with which we observe that while

$$\text{monochromaticity} \Rightarrow \text{perfect polarization } (P = 1)$$

the converse is not true; examples of the type

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 e^{i\delta} \end{pmatrix} f(t) \;:\; \text{perfectly correlated but non-monochromatic}$$

serve to establish that important point.

It is by incoherently *weighted superposition*—formally: by *formation of the density matrix*—that one gains access to the interior of the Poincaré sphere, where the arguments of §5 and the more general arguments of §9 tell us we in the general case want to be. Let

$$\begin{aligned}&\{\hat{\boldsymbol{s}}_1, \hat{\boldsymbol{s}}_2, \ldots, \hat{\boldsymbol{s}}_n\} \text{ describe points sprinked on the Poincaré sphere}\\ &\{p_1, p_2, \ldots, p_n\} \text{ describe their respective weights } (\, p_i \geqslant 0, \textstyle\sum p_i = 1)\end{aligned}$$

and observe that

$$\boldsymbol{s}_{\text{mean}} \equiv \sum p_i \hat{\boldsymbol{s}}_i \tag{137}$$

serves except in trivial cases (which is to say: except when all but one of the p_i vanish, which entails $\sum p_i^2 = 1$) to describe an *interior point* of the Poincaré sphere. Observe also that each interior point $\boldsymbol{s}_{\text{mean}}$ can be associated with infinitely many distinct “weighted sprinkles” $\sum p_i \hat{\boldsymbol{s}}_i$. Finally, form the density matrix

$$\boldsymbol{\rho} \equiv \sum p_i \mathsf{P}_i = \tfrac{1}{2}\begin{pmatrix} 1 + s_1 & s_2 - is_3 \\ s_2 + is_3 & 1 - s_1 \end{pmatrix}_{\text{mean}} \tag{138}$$

and observe once again that

- $\mathrm{tr}\boldsymbol{\rho}^2 \leqslant 1$ according as $\boldsymbol{s}_{\text{mean}}$ lies *interior to* or *on the surface of* the Poincaré sphere;
- $\boldsymbol{\rho}$ is hermitian in all cases, but projective ($\boldsymbol{\rho}^2 = \boldsymbol{\rho}$) if and only if $\mathrm{tr}\boldsymbol{\rho}^2 = 1$.

The condition

$$\mathrm{tr}\boldsymbol{\rho}^2 \leqslant 1 \text{ distinguishes } \begin{cases} \text{statistical "mixtures" of states} \\ \text{from "pure states" in quantum mechanics} \\ \\ \text{"partially polarized" optical beams } (0 \leqslant P < 1) \\ \text{from "100\% polarized" beams } (P = 1) \end{cases}$$

but the confluence of formalism makes it all the more important to recognize some interpretive distinctions:

- In quantum mechanics p_i refers to the statistical weight with which a state $|i)$ has been admixed into an ensemble of distinct quantum systems; within each such system one speaks of the *superposition* of states, but within the ensemble one speaks of *admixture*;
- In the optical theory of beams p_i refers to the *relative intensity*

$$p_i = \frac{\text{intensity of } i^{\text{th}} \text{ component}}{\text{total intensity}} = \frac{S_{0i}}{S_{01} + S_{02} + \cdots + S_{03}}$$

  of one component in a beam population which has been *incoherently but physically superimposed*. In optical theory the "ensemble" is absent, while in quantum theory the notion of "incoherent superposition" is absent. . . for the reason that the $\psi(x, t)$ supplied by quantum theory is not interpretable as a "signal;" one cannot "sit at x and monitor the changing value of ψ."

Stokes' "Principle of Optical Equivalence"—according to which[50]

> (alternatively assembled) optical beams which share identical Stokes' parameters cannot be distinguished by devices which look only to Stoke's parameters (and not to higher-order correlational moments)

—is seen now to be not quite so circular as it might at first sight appear; it asserts that (higher-order subtleties aside)

> distinct beam mixtures (incoherent superpositions) are, if described by identical density matrices $\boldsymbol{\rho}$, indistinguisable

which is surprising and useful information not only from an observational point of view, but also mathematically. The mechanism by which Stokes' principle comes to lie at the base of the (131) and (135)—two alternative "canonical representations" of $\boldsymbol{\rho}$—is developed in the caption to the following figure.[51]

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[50] See again p. 26.

[51] I am indebted to Thomas Wieting for my first exposure to this pretty fusion of geometry and alegebra.

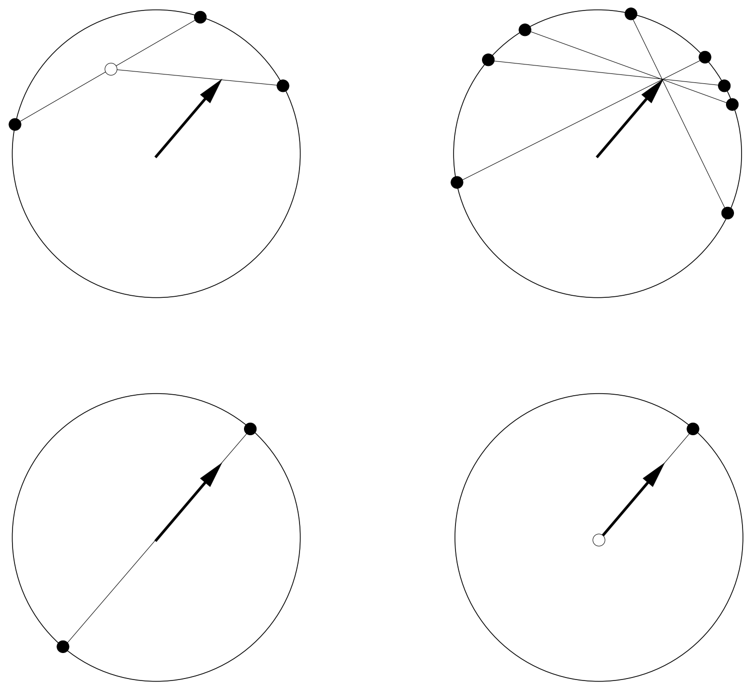

FIGURE 7: *Four different beam mixtures which yield the same* $\boldsymbol{s}_{\text{mean}}$ *(which yield, that is to say, the same density matrix* $\boldsymbol{\rho}$*) and therefore are, by the Principle of Optical Equivalence, indistinguishable. The circles are cross sections of the Poincaré sphere, and* $\boldsymbol{s}_{\text{mean}}$ *has been represented by an arrow. The figure at upper left illustrates how "Caulder's construction" can be used to locate* $\boldsymbol{s}_{\text{mean}}$ *for a 3-state mixture. The figure at upper right shows several equivalent 2-state mixtures. Elements of the 2-state mixture shown at lower left are diametric, therefore orthogonal; the figure illustrates the special nature of the "spectral representation" (131) of* $\boldsymbol{\rho}$*; taking* $\boldsymbol{s}_{\text{mean}}$ *to have length* P*, it becomes clear by the "teeter-totter principle" that the arrow points toward a state with weight* $\frac{1}{2}(1+P)$*, and from a state with weight* $\frac{1}{2}(1-P)$*. The distinguishing feature of the figure at lower right—designed to illustrate the mechanism responsible for (135)—is that one of the beams lies at the (unpolarized) origin; by the teeter-totter principle it has weight* $1-P$*.*

PART II: APPLICATIONS TO CLASSICAL MECHANICS

12. Isotropic oscillator: fundamentals. Look, by way of preparation, to the 1-dimensional case: To announce our interest in the classical physics of a harmonic oscillator we might write

$$H(x,p) = \tfrac{1}{2m}\big\{p^2 + (m\omega x)^2\big\} \tag{139}$$

giving

$$\left.\begin{aligned} \dot{x} &= -[H,x] = +p/m \\ \dot{p} &= -[H,p] = -m\omega^2 x \end{aligned}\right\} \tag{140}$$

whence

$$\ddot{x} + \omega^2 x = 0 \tag{141}$$

where the Poisson bracket has been defined in the usual way:

$$[A,B] \equiv \tfrac{\partial A}{\partial x}\tfrac{\partial B}{\partial p} - \tfrac{\partial B}{\partial x}\tfrac{\partial A}{\partial p} \tag{142}$$

To emphasize the quadratic structure of the Hamiltonian it becomes natural to write

$$H = \tfrac{1}{2}\lambda\big\{q^2 + y^2\big\} \quad \text{with} \quad \left\{\begin{aligned} y &\equiv \sqrt{\tfrac{m\omega^2}{\lambda}}\,x \\ q &\equiv \sqrt{\tfrac{1}{m\lambda}}\,p \end{aligned}\right. \tag{143}$$

where, it should be noticed, y and q are now *co-dimensional*. In this notation the canonical equations of motion (140) are found to read

$$\left.\begin{aligned} \dot{y} &= +\omega q = -\tfrac{\omega}{\lambda}[H,y] \\ \dot{q} &= -\omega y = -\tfrac{\omega}{\lambda}[H,q] \end{aligned}\right\} \tag{144}$$

where the brackets on the right have adquired now an unanticipated factor because the transformation $\big\{x,p\big\} \to \big\{y,q\big\}$ is *not canonical*:[52]

$$[y,q] = \tfrac{\omega}{\lambda}[x,p]$$

The striking structure of the transformed Hamiltonian (143) makes it difficult to resist writing

$$H = \lambda u^* u \quad \text{with} \quad \left\{\begin{aligned} u\ &\equiv \tfrac{1}{\sqrt{2}}(y+iq) \\ u^* &\equiv \text{complex conjugate} \end{aligned}\right.$$

---

[52] The transformation is, more precisely, *not symplectic*, but "canonical with a non-unit multiplier." The "multiplier" concept is absent from most elementary accounts of the theory of canonical transformations, but see A. Wintner, *Analytical Foundations of Celestial Mechanics* (1947), pp. 22–28. I introduce the notion at p. 7 in Chapter 7 of CLASSICAL DYNAMICS (1964), and treat the transformation properties of the Poisson bracket at p. 20.

In this notation the coupled equations of motion (143) decouple, becoming

$$\begin{aligned}\dot{u}\;\; &= -i\omega\, u \;\;= i\tfrac{\omega}{\lambda}\Big\{\tfrac{\partial H}{\partial u}\tfrac{\partial u}{\partial u^*} - \tfrac{\partial u}{\partial u}\tfrac{\partial H}{\partial u^*}\Big\} = -\tfrac{\omega}{i\lambda}[H, u\;\,]\\ \dot{u}^* &= +i\omega\, u^* = i\tfrac{\omega}{\lambda}\Big\{\tfrac{\partial H}{\partial u}\tfrac{\partial u^*}{\partial u^*} - \tfrac{\partial u^*}{\partial u}\tfrac{\partial H}{\partial u^*}\Big\} = -\tfrac{\omega}{i\lambda}[H, u^*]\end{aligned}$$

where the prefactor on the right arises (as before) from the circumstance that the transformation $\{x, y\} \to \{u, u^*\}$ is canonical with a non-unit multiplier, though the multiplier has become now *imaginary*:

$$[u, u^*] = \tfrac{\omega}{i\lambda}[x, p]$$

If λ is dimensionally an energy then u and u^* are dimensionless, but the classical theory supplies no "natural energy." If, however, (anticipating a practice standard to the quantum theory of oscillators) we allow ourselves to write

$\lambda = \hbar\omega$, where $\hbar$ has arbitrary value but dimensionality of "action"

and make the notational adjustment $u \mapsto a$ to emphasize that we have done so, then we have these summary formulæ

$$H = \hbar\omega\, a^*a \quad \text{with} \quad \begin{cases} a \;\;= \frac{1}{\sqrt{2}}\big\{\sqrt{m\omega/\hbar}\cdot x + i\sqrt{1/m\omega\hbar}\cdot p\big\}\\ a^* = \text{complex conjugate}\end{cases} \tag{145}$$

$$[a, a^*] = \tfrac{1}{i\hbar} \tag{146}$$

$$\left.\begin{aligned}\dot{a}\;\; &= -[H, a\;\,] = -\hbar\omega[a^*a, a\;\,] = -i\omega a\\ \dot{a}^* &= -[H, a^*] = -\hbar\omega[a^*a, a^*] = +i\omega a^*\end{aligned}\right\} \tag{147}$$

where all brackets are understood to be $[\bullet, \bullet]_{xp}$ brackets, as described at (142). The solution of (147)

$$a(t) = \mathcal{A}\, e^{-i(\omega t - \alpha)} \tag{148}$$

serve in effect to install a "clock" at the origin of the complex a-plane, and to inscribe an ellipse on the physical phase plane:

$$\left.\begin{aligned}x(t) &= +\mathcal{X}\cos(\omega t - \alpha) \quad \text{with} \quad \mathcal{X} \equiv \mathcal{A}\sqrt{2\hbar/m\omega}\\ p(t) &= -\mathcal{P}\,\sin(\omega t - \alpha) \quad \text{with} \quad \mathcal{P} \equiv \mathcal{A}\sqrt{2\hbar m\omega}\end{aligned}\right\} \tag{149}$$

Turning now to the isotropic 2-dimensional system

$$H(x, p) = \tfrac{1}{2m}\big\{(p_1^2 + p_2^2) + m^2\omega^2(x_1^2 + x_2^2)\big\} \tag{150.1}$$

in which we are at present primarily interested, we are led similarly to write

$$H = \tfrac{1}{2}\hbar\omega\big\{q_1^2 + q_2^2 + y_1^2 + y_2^2\big\} \tag{150.2}$$

$$= \hbar\omega\big\{a_1^*a_1 + a_2^*a_2\big\} \tag{150.3}$$

with a_1 and a_2 defined in direct imitation of (145):

$$\left.\begin{aligned} a_1 &\equiv \tfrac{1}{\sqrt{2}}\big\{\sqrt{m\omega/\hbar}\cdot x_1 + i\sqrt{1/m\omega\hbar}\cdot p_1\big\} \equiv \tfrac{1}{\sqrt{2}}\big\{y_1 + iq_1\big\} \\ a_2 &\equiv \tfrac{1}{\sqrt{2}}\big\{\sqrt{m\omega/\hbar}\cdot x_2 + i\sqrt{1/m\omega\hbar}\cdot p_2\big\} \equiv \tfrac{1}{\sqrt{2}}\big\{y_2 + iq_2\big\} \end{aligned}\right\} \tag{151}$$

Then

$$[a_1, a_1^*] = [a_2, a_2^*] = \tfrac{1}{i\hbar} \tag{152}$$

while $[a_1 \text{ or } a_1^*, a_2 \text{ or } a_2^*] = 0$. From the equations of motion

$$\left.\begin{aligned} \dot{a}_1 &= -[H, a_1] = -i\omega a_1 \\ \dot{a}_2 &= -[H, a_2] = -i\omega a_2 \end{aligned}\right\} \tag{153}$$

and their conjugates[53] one obtains

$$\left.\begin{aligned} a_1(t) &= \mathcal{A}_1 e^{-i\,\omega t} \\ a_2(t) &= \mathcal{A}_2 e^{-i(\omega t+\delta)} \end{aligned}\right\} \tag{154}$$

which in (rescaled) physical variables read

$$\left.\begin{aligned} y_1(t) &= +\sqrt{2}\mathcal{A}_1 \cos(\omega t) \\ q_1(t) &= -\sqrt{2}\mathcal{A}_1 \sin(\omega t) \\ y_2(t) &= +\sqrt{2}\mathcal{A}_2 \cos(\omega t+\delta) \\ q_2(t) &= -\sqrt{2}\mathcal{A}_2 \sin(\omega t+\delta) \end{aligned}\right\} \tag{155}$$

The equation

$$H(x_1, x_2, p_1, p_2) = E \tag{156}$$

serves to identify an E-parameterized family of nested *surfaces of constant energy* in 4-dimensional phase space. From (150) we see that those surfaces are

- hyperellipsoidal in the physical variables $\{x_1, p_1, x_2, p_2\}$, but become
- hyperspherical in the rescaled variables $\{y_1, q_1, y_2, q_2\}$, and

and are (in a manner of speaking) circular in 2-dimensional "complex phase space." The equations (155) trace and periodically retrace a t-parameterized curve on the hypersphere of squared radius $2(\mathcal{A}_1^2+\mathcal{A}_2^2)$, with which we associate the energy

$$E = \hbar\omega(\mathcal{A}_1^2 + \mathcal{A}_2^2) \tag{157}$$

Several typical projections of that curve are shown in Figure 8. Superposition of the projections onto the $\{y_1, q_1\}$ and $\{y_2, q_2\}$ planes yields a pair of concentric circles with $\circlearrowright$ chirality; reinstallation of physical coordinates $\{x_1, p_1\}$ and $\{x_2, p_2\}$ converts those into a concentric pair of identically figured ellipses in standard position (principal axes coincident with the coordinate axes). On the other hand, superposition of the projections onto the $\{y_1, y_2\}$ and $\{q_1, q_2\}$ planes yields duplicate copies of an ellipse the size/orientation/figure/chirality of which is case-dependent; reinstallation of physical coordinates yields a pair of concentric ellipses which differ only with respect to size.

[53] This little phrase will henceforth be taken for granted.

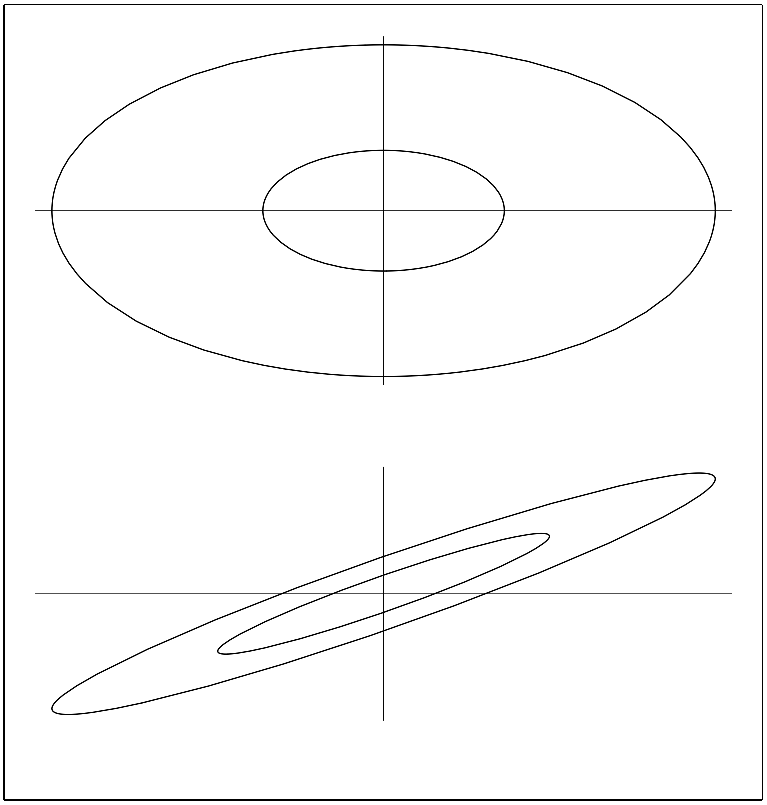

FIGURE 8: *Four projections of a curve that lives in 4-dimensional phase space. Shown above are superimposed plots of the curves* $\{x_1(t), p_1(t)\}$ *and* $\{x_2(t), p_2(t)\}$*; both ellipses have* ↻ *chirality, and in all cases their coincident principal axes are aligned with the coordinate axes. Shown below are superimposed plots of the curves* $\{x_1(t), x_2(t)\}$ *and* $\{p_1(t), p_2(t)\}$*; chirality and other characteristics of those concentric ellipses are case-dependent.*

In particular, we have

$$
\begin{aligned}
\boldsymbol{x}(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad (158)\\
&\qquad x_1(t) = \mathcal{X}_1 \cos(\omega t) \qquad \text{with} \quad \mathcal{X}_1 = \sqrt{2\hbar/m\omega}\,\mathcal{A}_1 \\
&\qquad x_2(t) = \mathcal{X}_2 \cos(\omega t + \delta) \quad \text{with} \quad \mathcal{X}_2 = \sqrt{2\hbar/m\omega}\,\mathcal{A}_2
\end{aligned}
$$

which serve to describe the literal motion of the bob within its 2-dimensional configuration space. Since (158) is structurally identical to (2) we can, by direct appropriation of (20), introduce "mechanical Stokes parameters"

$$\left.\begin{aligned} S_0 &= \mathcal{X}_1^2 + \mathcal{X}_2^2 \\ S_1 &= \mathcal{X}_1^2 - \mathcal{X}_2^2 &&= S_0\cos 2\chi\cos 2\psi \\ S_2 &= 2\mathcal{X}_1\mathcal{X}_2\cos\delta &&= S_0\cos 2\chi\sin 2\psi \\ S_3 &= 2\mathcal{X}_1\mathcal{X}_2\sin\delta &&= S_0\sin 2\chi \end{aligned}\right\} \tag{159}$$

to describe the figure and chiral sense of the orbit.

But $\boldsymbol{x}(t)$ moves subject to principles of mechanics in a sense much more immediate than can be said of the electrical field vector $\mathbf{E}(t)$. The details of the mathematical mechanism by which $H(\boldsymbol{x}(t),\boldsymbol{p}(t))$ becomes t-independent are typically quite intricate, but for the isotropic oscillator are uniquely transparent; we (by (150.3)) have

$$H = \hbar\omega\,\boldsymbol{a}^\dagger\boldsymbol{a} \quad\text{with}\quad \boldsymbol{a}\equiv\begin{pmatrix}a_1\\a_2\end{pmatrix} \tag{160}$$

and when we introduce (154)—notated

$$\boldsymbol{a}(t) = \boldsymbol{A}\,e^{-i\omega t} \tag{161}$$

—we find that the exponentials cancel, leaving behind an expression which is trivially/manifestly t-independent. The argument serves, in fact, to establish the t-independence of *every* expression of the form[54]

$$Q = \boldsymbol{a}^\dagger\mathbb{Q}\boldsymbol{a}$$

where reality entails the hermiticity of $\mathbb{Q}$. We find it natural, therefore, to introduce (compare (53))

$$\left.\begin{aligned} Q_0 &= \boldsymbol{a}^\dagger\,\mathbb{s}_0\,\boldsymbol{a} \quad\text{with (as before)}\quad &\mathbb{s}_0 &\equiv \begin{pmatrix}1&0\\0&1\end{pmatrix}\\ Q_1 &= \boldsymbol{a}^\dagger\,\mathbb{s}_1\,\boldsymbol{a} \quad\text{with} &\mathbb{s}_1 &\equiv \begin{pmatrix}1&0\\0&-1\end{pmatrix}\\ Q_2 &= \boldsymbol{a}^\dagger\,\mathbb{s}_2\,\boldsymbol{a} \quad\text{with} &\mathbb{s}_2 &\equiv \begin{pmatrix}0&1\\1&0\end{pmatrix}\\ Q_3 &= \boldsymbol{a}^\dagger\,\mathbb{s}_3\,\boldsymbol{a} \quad\text{with} &\mathbb{s}_3 &\equiv \begin{pmatrix}0&-i\\i&0\end{pmatrix} \end{aligned}\right\} \tag{162}$$

[54] Here I find it natural to preserve quadraticity, but that is inessential; one has automatic t-independence for every expression of the form

$$Q = (\text{arbitrary function of } a_1 \text{ and } a_2)(\text{its conjugate})$$

We have here a quartet of manifestly conserved observables, of which (drawing upon (151))

$$
\begin{aligned}
\hbar\omega Q_0 &= H_1 + H_2 && (163.0)\\
\hbar\omega Q_1 &= H_1 - H_2 && (163.1)\\
&\qquad H_1 \equiv \hbar\omega a_1^* a_1 = \tfrac{1}{2}\hbar\omega(q_1^2 + y_1^2) = \tfrac{1}{2m}(p_1^2 + m^2\omega^2 x_1^2)\\
&\qquad H_2 \equiv \hbar\omega a_2^* a_2 = \text{etc.}\\
\hbar\omega Q_2 &= \hbar\omega(a_1^* a_2 + a_2^* a_1)\\
&= \hbar\omega(y_1 y_2 + q_1 q_2)\\
&= \tfrac{1}{m}(p_1 p_2 + m^2\omega^2 x_1 x_2) && (163.2)\\
\hbar\omega Q_3 &= i\hbar\omega(a_2^* a_1 - a_1^* a_2)\\
&= \hbar\omega(y_1 q_2 - y_2 q_1)\\
&= \ \ \omega(x_1 p_2 - x_2 p_1) && (163.3)
\end{aligned}
$$

provide descriptions in terms of (rescaled) physical variables. The observables H_1 and H_2 are, in an obvious sense, "partial Hamiltonians" of the system. Evidently

$$
\begin{aligned}
\hbar\omega Q_0 &= \text{Hamiltonian}\\
\hbar\omega Q_1 &= \text{“Hamiltonian difference”}\\
\hbar\omega Q_2 &= \quad ?\\
\hbar\omega Q_3 &= \omega\cdot(\text{angular momentum})
\end{aligned}
$$

To establish that—as we anticipate—$\{Q_1, Q_2, Q_3\}$ comprise more of a "natural package" than might, on this evidence, appear, we observe that in easy consequence of the Poisson bracket relations

$$[a_r, a_s^*] = \tfrac{1}{i\hbar}\delta_{rs}$$

we have

$$[\boldsymbol{a}^\dagger\mathbb{Q}\,\boldsymbol{a}, \boldsymbol{a}^\dagger\mathbb{R}\,\boldsymbol{a}] = \tfrac{1}{i\hbar}\boldsymbol{a}^\dagger[\mathbb{Q},\mathbb{R}]\boldsymbol{a} \ : \ \text{any } 2\times 2 \text{ matrices } \mathbb{Q} \text{ and } \mathbb{R} \tag{164}$$

From the familiar commutation properties (55) of the Pauli matrices it follows therefore that

$$
\begin{aligned}
[Q_1, Q_2] &= \tfrac{2}{\hbar}Q_3\\
[Q_2, Q_3] &= \tfrac{2}{\hbar}Q_1\\
[Q_3, Q_1] &= \tfrac{2}{\hbar}Q_2
\end{aligned}
$$

The observables

$$
\left.
\begin{aligned}
L_1 &\equiv \tfrac{1}{2}\hbar Q_1 = \tfrac{1}{4m\omega}(p_1^2 + m^2\omega^2 x_1^2) - \tfrac{1}{4m\omega}(p_2^2 + m^2\omega^2 x_2^2)\\
L_2 &\equiv \tfrac{1}{2}\hbar Q_2 = \tfrac{1}{2m\omega}(p_1 p_2 + m^2\omega^2 x_1 x_2)\\
L_3 &\equiv \tfrac{1}{2}\hbar Q_3 = \tfrac{1}{2}(x_1 p_2 - x_2 p_1)
\end{aligned}
\right\} \tag{165}
$$

therefore satisfy Poisson bracket relations

$$\left.\begin{aligned}[][L_1,L_2]&=L_3\\ [L_2,L_3]&=L_1\\ [L_3,L_1]&=L_2\end{aligned}\right\}\tag{166}$$

familiar from the classical theory of angular momentum—relations well known to be characteristic of the generators of the 3-dimensional rotation group $O(3)$.

To remark that the infinitesimial unitary transformation

$$\boldsymbol{a}\longrightarrow\boldsymbol{a}+id\theta\big(\lambda_1\mathbb{S}_1+\lambda_2\mathbb{S}_2+\lambda_3\mathbb{S}_3\big)\boldsymbol{a}\tag{167.1}$$

induces an infinitesimal rotation amongst the Q's

$$\begin{aligned}\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}&=\begin{pmatrix}\boldsymbol{a}^\dagger\mathbb{S}_1\boldsymbol{a}\\\boldsymbol{a}^\dagger\mathbb{S}_2\boldsymbol{a}\\\boldsymbol{a}^\dagger\mathbb{S}_2\boldsymbol{a}\end{pmatrix}\\ &\downarrow\\ \begin{pmatrix}Q_1+dQ_1\\Q_2+dQ_2\\Q_3+dQ_3\end{pmatrix}&=\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}+id\theta\,\boldsymbol{a}^\dagger\begin{pmatrix}\lambda_1[\mathbb{S}_1,\mathbb{S}_1]+\lambda_2[\mathbb{S}_1,\mathbb{S}_2]+\lambda_3[\mathbb{S}_1,\mathbb{S}_3]\\\lambda_1[\mathbb{S}_2,\mathbb{S}_1]+\lambda_2[\mathbb{S}_2,\mathbb{S}_2]+\lambda_3[\mathbb{S}_2,\mathbb{S}_3]\\\lambda_1[\mathbb{S}_3,\mathbb{S}_1]+\lambda_2[\mathbb{S}_3,\mathbb{S}_2]+\lambda_3[\mathbb{S}_3,\mathbb{S}_3]\end{pmatrix}\boldsymbol{a}\\ &=\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}-2id\theta\begin{pmatrix}0&\lambda_2&-\lambda_3\\-\lambda_2&0&\lambda_1\\\lambda_1&-\lambda_1&0\end{pmatrix}\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}\end{aligned}\tag{167.2}$$

is to remark simply that oscillator physics has led us back again to a familiar mathematical intersection. . . but this time with a twist: the transformations (167) are *canonical in 4-dimensional phase space*, Lie-generated by

$$G=\hbar(\lambda_1Q_1+\lambda_2Q_2+\lambda_3Q_3)\tag{168}$$

The argument hinges on the observation[55] that

$$[\boldsymbol{a}^\dagger\mathbb{Q}\,\boldsymbol{a},\,\boldsymbol{a}]=\tfrac{i}{\hbar}\mathbb{Q}\,\boldsymbol{a}\tag{169}$$

from which (164) can be recovered as a corollary, and which entails

$$[G,\boldsymbol{a}]=i\big(\lambda_1\mathbb{S}_1+\lambda_2\mathbb{S}_2+\lambda_3\mathbb{S}_3\big)\boldsymbol{a}\tag{170}$$

The reconstruction of (167.2) is now so straightforward (but fun!) that I omit the details. While mechanics supplies variables $\boldsymbol{p}$ conjugate to the variables $\boldsymbol{x}$,

---

[55] Write

$$[\boldsymbol{a}^\dagger\mathbb{Q}\,\boldsymbol{a},a_r]=\sum\sum Q_{pq}[a_p^*a_q,a_r]=-\sum\sum Q_{pq}\tfrac{1}{i\hbar}\delta_{pr}a_q=\tfrac{i}{\hbar}\sum Q_{rs}a_s$$

and posits the dynamical problem in phase space (where the theory of canonical transformations resides), electrodynamics supplies no variables conjugate to $\mathbf{E}$, no “phase space,” no “canonical transformations;” it is in that sense that oscillator mechanics has added a new dimension—a “novel twist”—to the nest of ideas introduced into optics by Stokes/Poincaré/Jones.

I would emphasize the the Q’s are by nature *observables—functions*, to be distinguished from their valuations just as, and for the same reason that, we are careful to distinguish the Hamiltonian $H(\boldsymbol{x},\boldsymbol{p})$ from its valuation E. From

$$[H,Q_r] = \hbar\omega\cdot[Q_0,Q_r] = \hbar\omega\cdot\tfrac{1}{i\hbar}\boldsymbol{a}^\dagger[\mathbb{s}_o,\mathbb{s}_r]\,\boldsymbol{a} = 0 \tag{171}$$

we learn that the Q’s are *conserved* observables (constants of the dynmical motion), while from

$$\begin{aligned} Q_0^2 - Q_1^2 - Q_2^2 - Q_3^2 = (a_1^*a_1 + a_2^*a_2)^2 &- (a_1^*a_1 - a_2^*a_2)^2 \\ &- (a_1^*a_2 + a_2^*a_1)^2 \\ &+ (a_1^*a_2 - a_2^*a_2)^2 \\ = 0 \end{aligned} \tag{172}$$

we learn that that Q’s are not independent, but subject to a familiar constraint.

The “mechanical Stokes parameters” introduced at (159) are dimensionally “squared lengths,” and arise when one looks to the conserved *value* of the (dimensionless) observables Q:

$$S_\mu = (2\hbar/m\omega)\cdot(\text{numerical value of } Q_\mu)$$

13. The driven/damped isotropic oscillator. The isotropic oscillator has been seen to present us with a theory which is formally homologous to the theory of monochromatic optical beams, and which reveals an especially close relationship to the Jones formalism. The Mueller/Jones calculi are, however, concerned more with “beam manipulation” than with “beam description;” they can, I suppose, be adapted to the theory of oscillators, but the theory of “kicked oscillators”—analogs of beams intercepted by devices—makes no strong claim to our attention. The orbital precession which results from the presence of anharmonicity is (more interestingly) directly analogous to “optical activity,” but that also is optico-mechanics which I must on this occasion be content to pass by. Such formal analogies are interesting, but their practical utility tends to be diminished by this physical circumstance: at optical frequencies one cannot “watch” the motion of $\mathbf{E}(t)$, but one can quite feasibly watch the motion of $\boldsymbol{x}(t)$. The optician and the mechanic find themselves in fundamentally different situations, and in some respects it becomes artificial for one to borrow the tools of the other. On such grounds one expects the optico-mechanical analogy to acquire deepened interest when looks to the *quantum* theory of the isotropic oscillator. Of the several things that optics has to say to classical mechanics, I am motivated to look here only to one.

To describe a driven isotropic oscillator we write

$$\left.\begin{aligned} \ddot{x}_1 &= -\omega^2 x_1 + f_1(t) \\ \ddot{x}_2 &= -\omega^2 x_2 + f_2(t) \end{aligned}\right\} \tag{173}$$

Which is to say: we write (compare(150.1))

$$H(x,p) = \tfrac{1}{2m}\big\{(p_1^2 + p_2^2) + m^2\omega^2(x_1^2 + x_2^2)\big\} - m\big\{x_1 f_1(t) + x_2 f_2(t)\big\} \tag{174}$$

giving

$$\begin{aligned} \dot{x}_1 &= \tfrac{1}{m}p_1 \\ \dot{p}_1 &= -m\omega^2 x_1 + mf_1(t) \\ \dot{x}_2 &= \tfrac{1}{m}p_2 \\ \dot{p}_2 &= -m\omega^2 x_2 + mf_2(t) \end{aligned}$$

from which (173) can be recovered. In the notation introduced at (145) we therefore have

$$\dot{\boldsymbol{a}} = -i\omega\,\boldsymbol{a} + i\boldsymbol{g}(t) \tag{175}$$

with $\boldsymbol{g} \equiv \sqrt{m/2\hbar\omega}\,\boldsymbol{f}$. If $\boldsymbol{A} \equiv \boldsymbol{a}(0)$ is prescribed, then the solution of (175) can be described

$$\boldsymbol{a}(t) = e^{-i\omega t}\Big\{\boldsymbol{A} + i\int_0^t e^{i\omega s}\boldsymbol{g}(s)\,ds\Big\} \tag{176}$$

To accommodate damping—mechanical analog of beam progress through an absorbtive medium—we might proceed phenomenologically, writing

$$(173) \quad\longrightarrow\quad \begin{cases} \ddot{x}_1 = -2\gamma\dot{x}_1 - \omega^2 x_1 + f_1(t) \\ \ddot{x}_2 = -2\gamma\dot{x}_2 - \omega^2 x_2 + f_2(t) \end{cases}$$

or perhaps (and even though such equations clearly fail the canonicity test

$$\partial\dot{x}_i/\partial x_j = \partial^2 H/\partial p_i\partial x_j = -\partial\dot{p}_j/\partial p_i$$

and therefore cannot be obtained from a Hamiltonian)

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \tfrac{1}{m}\boldsymbol{p} \\ \dot{\boldsymbol{p}} &= -2\gamma\boldsymbol{p} - m\omega^2\boldsymbol{x} + m\boldsymbol{f} \end{aligned}\right\} \tag{177}$$

which can also be expressed

$$\left.\begin{aligned} \dot{\boldsymbol{a}} \; &= -(\gamma + i\omega)\boldsymbol{a} \;\, + \gamma\boldsymbol{a}^* + i\boldsymbol{g} \\ \dot{\boldsymbol{a}}^* &= -(\gamma - i\omega)\boldsymbol{a}^* + \gamma\boldsymbol{a} \;\, - i\boldsymbol{g} \end{aligned}\right\} \tag{178}$$

with $\boldsymbol{g}(t) \equiv \sqrt{m/2\hbar\omega}\,\boldsymbol{f}(t)$. From the coupled system (178) it follows that (in the absence of forcing: $\boldsymbol{g} = \boldsymbol{0}$) energy dies exponentially, but with a gurgle[56]

$$\dot{E} = -2\gamma E + \underbrace{\gamma\hbar\omega(\boldsymbol{a}^*\!\cdot\boldsymbol{a}^* + \boldsymbol{a}\cdot\boldsymbol{a})}_{\text{“gurgle”}}$$

A cleaner theory results if in place of (178) we write the uncoupled system

$$\left.\begin{aligned} \dot{\boldsymbol{a}} &= -(\gamma + i\omega)\boldsymbol{a} \ + i\boldsymbol{g} \\ \dot{\boldsymbol{a}}^* &= -(\gamma - i\omega)\boldsymbol{a}^* - i\boldsymbol{g} \end{aligned}\right\} \tag{179}$$

—which step we might attempt to justify on the grounds that

- the uncoupled system is easier to solve (one has only to adjust (176));
- energy dies more comfortably (no gurgle: $\dot{E} = -2\gamma E$);
- the theory is only intended to be “phenomenological” anyway.

The point is that—whether one works from (176) or from some more elaborate variant—the installation of driving forces and/or damping induces

$$\boldsymbol{a}(t) = e^{-i\omega t}\boldsymbol{A} \quad\longrightarrow\quad \boldsymbol{a}(t) = e^{-i\omega t}\boldsymbol{A}(t)$$

and lends t-dependence to the associated “mechanical Stokes parameters”

$$S_\mu \quad\longrightarrow\quad S_\mu(t)$$

We are in position, therefore, to bring optical language and methods (partial polarization, coherence/correlation) to the description of “tickled isotropic oscillators” (oscillators in weak magnetic fields, oscillators subject to stochastic perturbation, etc.). And to bring notions borrowed from oscillator theory (resonance) to the description of optical phenomena (laser beam production by stimulated emission). I shall, on this occasion, venture down none of those side trails, though they appear to be fairly easy hikes.

14. Fundamentals of the 2-dimensional Kepler problem. Bertrand's theorem[57] asserts that precisely two central potentials

- the isotropic oscillator potential $U(r) = \frac{1}{2}kr^2$, and
- the Coulomb potential $U(r) = -k/r$ $(k > 0)$

have the property that *all bound orbits close upon themselves*. As it happens,

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[56] See ANALYTICAL METHODS OF PHYSICS (1981), p. 170.

[57] Joseph Bertrand (1873). See §3-6 and the detailed discussion which appears as Appendix A in H. Goldstein's *Classical Mechanics* (2nd edition 1980). Goldstein cites as his source H. C. Plummer, *An Introductory Treatise on Dynamical Astronomy* (1918, reprinted in 1960). For a useful sketch see §2.3.3 in J. V. José & E. J. Saletan, *Classical Dynamics* (1998), and for a much deeper and more modern account of what the author calls the “Bertrand-Königs theorem” see §§4.4 & 4.5 in J. L. McCauley, *Classical Mechanics* (1997).

the bound orbits are in both cases elliptical, though with this difference:
- the force center marks the *center* of the orbital ellipse in the former case;
- the force center marks *one focus* of the orbital ellipse in the letter case.

And it is from the former case to the latter that we now turn.

Central force motion lies necessarily in a plane, and classically we lose nothing (but gain in the elimination of some extraneous clutter) if we restrict our attention to the physics *written onto the orbital plane*.[58] We look, therefore, to the "2-dimensional Kepler problem"

$$H = \tfrac{1}{2m}(p_1^2 + p_2^2) - k(x_1^2 + x_2^2)^{-\frac{1}{2}} \tag{180}$$

Recently I have had occasion[59] to explore ramifications of a fact which came accidentally to my attention in quite another connection;[60] namely, that the problem thus posed is "separable in the sense of Liouville" in infinitely many distinct coordinate systems—the elliptic (or "confocal conic") coordinate systems which have
- one focus coincident with the force center
- the other (empty) focus positioned arbitrarily

and from which the more familiar polar and confocal parabolic coordinates can be recovered as limiting cases. Liouville's argument yields a separation constant which, when described in terms of phase coordinates $\{\boldsymbol{x}, \boldsymbol{p}\}$, acquires the status of a conserved observable

$$[H, G] = 0 \quad \text{with} \quad G = ma^2H + m\boldsymbol{a}\cdot\boldsymbol{K} + \tfrac{1}{2}L^2 \tag{181}$$

where $a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}$, $\boldsymbol{a}$ locates the "center" of the elliptic coordinate system (bisector of the line linking the occupied focus (origin) to the empty focus) and

$$L \equiv x_1p_2 - x_2p_1 \tag{182.1}$$

$$\left.\begin{aligned} K_1 &\equiv +\tfrac{1}{m}p_2L - kx_1\tfrac{1}{\sqrt{x_1^2+x_2^2}} \\ K_2 &\equiv -\tfrac{1}{m}p_1L - kx_2\tfrac{1}{\sqrt{x_1^2+x_2^2}} \end{aligned}\right\} \tag{182.2}$$

Since $\boldsymbol{a}$ is arbitrary, we are, in effect, supplied at (181) with a population of *three* conserved observables:

$$[H, L] = [H, K_1] = [H, K_2] = 0 \tag{183}$$

[58] In "Classical/quantum theory of 2-dimensional hydrogen" (Reed College Physics Seminar Notes 1999) I draw attention to the fact that such simplification is not permitted quantum mechanically; most conspicuous among its several defects is the fact that it leads to the wrong energy spectrum.

[59] "Reduced Kepler Problem in elliptic coordinates" (1998).

[60] "Kepler Problem by descent from the Euler Problem" (Physics Seminar Notes 1996).

The argument is pretty, but for the purposes immediately at hand superfluous; it is sufficient to remark that the constructions (182) and Keplerean conservation laws (183) are "well-known," as indeed they are: L is just the angular momentum (component normal to the orbital plane), and its conservation a reflection of the rotational symmetry of the system, while $\boldsymbol{K}$ is the celebrated "Runge-Lenz vector"[61] (component lying *in* the orbital plane), called into being as a conserved object by the fact of orbital closure. Equations (183) are susceptible to direct computational verification, so stand on their own legs, independently of the support of any elaborate theory, however "illuminating" such support may be. Direct computation reveals, moreover, that

$$\begin{aligned}
[L, K_1] &= K_2\\
[K_2, L] &= K_1\\
[K_1, K_2] &= -\tfrac{2}{m}H\cdot L
\end{aligned}$$

which expresses "algebraic closure" in the Poisson bracket sense. If, motivated by this result, we define

$$\left.\begin{aligned}
J_1 &\equiv K_1\Big/\sqrt{-\tfrac{2}{m}H}\\
J_2 &\equiv K_2\Big/\sqrt{-\tfrac{2}{m}H}\\
J_3 &\equiv L
\end{aligned}\right\}\qquad(184)$$

then we achieve

$$\left.\begin{aligned}
[J_3, J_1] &= J_2\\
[J_2, J_3] &= J_1\\
[J_1, J_2] &= J_3
\end{aligned}\right\}\qquad(185)$$

The observables $\{J_1, J_2, J_3\}$ are co-dimensional (each has the dimension of action), and within the bound sector of phase space each is real.

Each solution $\{\boldsymbol{x}(t), \boldsymbol{p}(t)\}$ draws a closed curve $\mathcal{C}$ in 4-dimensional phase space. Each such $\mathcal{C}$ is inscribed simultaneously on

- a 3-dimensional surface Σ_H of constant H;
- a 3-dimensional surface Σ_{J_1} of constant J_1;
- a 3-dimensional surface Σ_{J_2} of constant J_2;
- a 3-dimensional surface Σ_{J_3} of constant J_3.

But $\Sigma_H \cap \Sigma_{J_1} \cap \Sigma_{J_2} \cap \Sigma_{J_3}$ is a point (not a curve) unless the observables in

[61] For a good account of the basic theory see §3-9 in Goldstein's 2nd edition. In "Prehistory of the 'Runge-Lenz' vector" (AJP **43**, 737 (1975)) Goldstein traces the history of what he calls the "Laplace-Runge-Lenz vector" back to Laplace (1799). Reader response to that paper premitted him in a subsequent paper ("More on the prehistory of the Laplace-Runge-Lenz vector," AJP **44**, 1123 (1976)) to trace the idea back even further, to the work of one Jakob Hermann (1710) and its elaboration by Johann Bernoulli (1712). Goldstein's revised suggestion that $\boldsymbol{K}$ be called the "Hermann-Bernoulli-Laplace vector" seems unlikely to catch on, historical justice notwithstanding.

question are subject to a constraint (functionally interdependent)... and indeed: computation (executed on a hunch, and made feasible only with the assistance of *Mathematica*) gives

$$K_1^2 + K_2^2 - \tfrac{2}{m}H\cdot L^2 = k^2 \tag{186.1}$$

of which

$$H\cdot(J_1^2 + J_2^2 + J_3^2) = -\tfrac{1}{2}mk^2 \tag{186.2}$$

and

$$\begin{aligned} J_0^2 - J_1^2 - J_2^2 - J_3^2 &= 0 \\ J_0^2 &\equiv -\tfrac{1}{2}mk^2/H \end{aligned} \tag{186.3}$$

provide alternative formulations. The quantum analog of pretty result was noted (no small accomplishment!) and used to critical effect by Pauli in work to which I have several times alluded.[62] Notice that (186) holds meaningfully *throughout* phase space (not just in the bound sector), since it involves no expressions of the form $\sqrt{\text{negative}}$. From (186.1) we learn that

$$K^2 = k^2 + \tfrac{2}{m}E\ell^2 \tag{187}$$

where

$$\begin{aligned} K &\equiv \text{conserved numerical } \textit{magnitude} \text{ of the Lenz vector } \boldsymbol{K} \\ E &\equiv \text{energy (conserved numerical value of the Hamiltonian } H) \\ \ell &\equiv \text{angular momentum (conserved numerical value of } L) \end{aligned}$$

We learn, in other words, that the new information conveyed by $\boldsymbol{K}$ resides not in its magnitude (which is implicit in the values of E and ℓ) but in its *direction* (as indicated by $\hat{\boldsymbol{K}}$) which—familiarly, but as I now demonstrate—indicates the *orientation of the principal axis of the orbital ellipse*. Write

$$\boldsymbol{K} = \begin{pmatrix} K_1 \\ K_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{L} = \begin{pmatrix} 0 \\ 0 \\ \ell \end{pmatrix}$$

and observe that (182.2) can be notated

$$\boldsymbol{K} = \tfrac{1}{m}\,\boldsymbol{p}\times\boldsymbol{L} - \tfrac{k}{r}\,\boldsymbol{x} \tag{188}$$

Look—as a matter merely of computational convenience—to either of the two points at which the orbit crosses the principal axis (Figure 9). At such points it becomes *obvious* that $\boldsymbol{K}$ *runs parallel to the principal axis* (because $\boldsymbol{p}\times\boldsymbol{L}$ and $\boldsymbol{x}$ both do), which was the point at issue. Dotting (188) into itself, one is led back again (this time by the most elementary of means) to (187).

[62] See §3 in some previously cited Seminar Notes.[58]

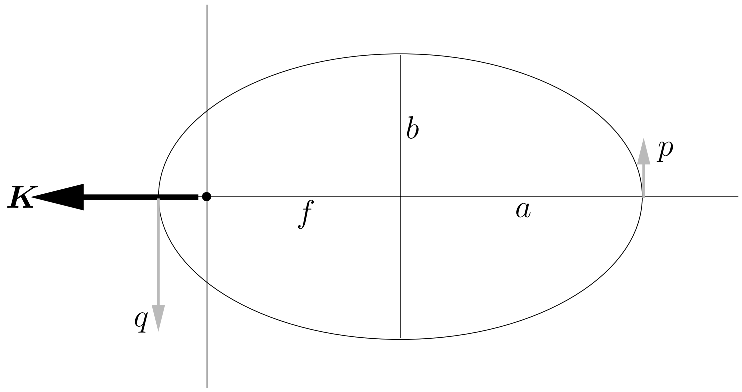

FIGURE 9: *Notation employed in the following discussion.*

Working from the figure, we have $\ell = (a+f)p = (a-f)q$ and

$$E = \tfrac{1}{2m}p^2 - \tfrac{k}{a+f} = \tfrac{1}{2m}q^2 - \tfrac{k}{a-f}$$

giving $E = \frac{1}{2m}\big(\frac{\ell}{a\pm f}\big)^2 - \frac{k}{a\pm f}$ whence

$$(a\pm f)^2 + \tfrac{k}{E}(a\pm f) - \tfrac{\ell^2}{2mE} = 0$$

Therefore

$$\begin{aligned} a\pm f &= -\tfrac{k}{2E} \pm \tfrac{1}{2}\sqrt{\big(\tfrac{k}{E}\big)^2 + \tfrac{2\ell^2}{mE}} \\ &= -\tfrac{k}{2E} \pm \tfrac{1}{2}\sqrt{\big(\tfrac{K}{E}\big)^2} \quad \text{by (187)} \end{aligned}$$

from which we conclude that

- the semi-major axis a depends only upon the energy (which for bound states is negative): $a = -\frac{k}{2E}$
- the focal length f depends upon E and ℓ, but upon the latter only as it enters into the construction (187) of K: $f = -\frac{K}{2E}$
- the semi-minor axis $b = \sqrt{a^2 - f^2}$ can be described $b = \sqrt{-\ell^2/2mE}$, which in the circular case $a = b$ entails $E = -mk^2/2\ell^2$.
- the ellipticity of the orbital ellipse can be described

$$\text{ellipticity } e \equiv f/a = K/k \tag{189}$$

- I assert in advance of proof (see the final paragraph of this section) that $\boldsymbol{K}$ is directed *toward the perihelion* (orbital point of closest approach).

The vector $\boldsymbol{p}(t)$ traces in momentum space a curve $\mathcal{H}$ (projectively related to $\mathcal{C}$) which was apparently first studied by Hamilton,[63] and was called by him

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[63] See Chapter 24 of T. L. Hankins' *Sir William Rowan Hamilton* (1980) and the second of the Goldstein papers cited above.[61] It was in connection with this work—inspired by the discovery of Neptune (1846)—that Hamilton was led to the independent reinvention of the "Hermann-. . . -Hamilton-. . . -Lenz vector."

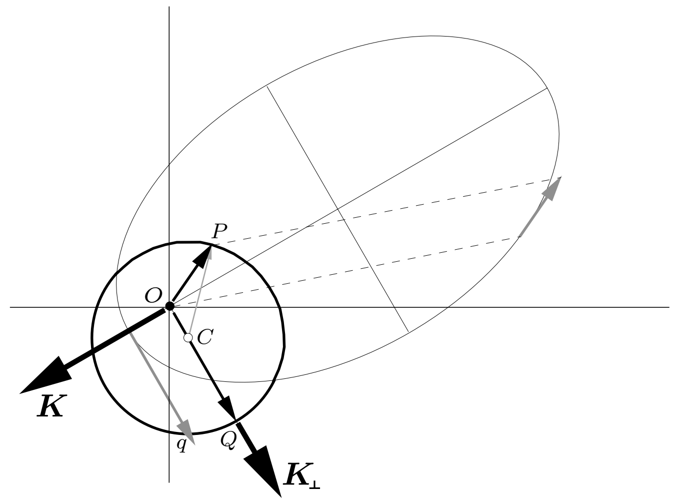

Figure 10: *Keplerean orbit* $\mathcal{G}$ *superimposed upon the hodograph* $\mathcal{H}$*. It was Hamilton's discovery that the Keplerean hodograph is circular, dentered on a line which stands normal to the principal axis at the force center.* Q *identifies the momentum at perihelion, and* q *the associated orbital tangent. The dogleg construction* $OP = OC + CP$ *illustrates the meaning of (190), and dashed lines indicate how points on the hodograph are to be associated with tangents to the orbit.*

the "hodograph." Working from (188), we have

$$\boldsymbol{K}_\perp \equiv \boldsymbol{L}\times\boldsymbol{K} = \tfrac{1}{m}\ell^2\boldsymbol{p} - \tfrac{k}{r}\boldsymbol{L}\times\boldsymbol{x}$$

giving

$$\begin{aligned}\boldsymbol{p} &= (m/\ell^2)\boldsymbol{K}_\perp + (mk/\ell^2)\,\boldsymbol{L}\times\hat{\boldsymbol{x}} \qquad (190)\\ &= (\text{constant vector of length } \tfrac{mK}{\ell})\\ &\qquad\qquad + (\text{vector that traces a circle of radius } \tfrac{mk}{\ell})\end{aligned}$$

From (187) it follows that $(\text{radius})^2 - (\text{displacement})^2 = -2mE > 0$: we are brought to the striking conclusion—illustrated in Figure 10, and apparently overlooked by Newton—that the Keplerean hodograph is *circular*, and envelops the origin (or doesn't) according as the spatial orbit is bound (or or unbound).

While projection of $\mathcal{C}$ onto momentum space yields the relatively unfamiliar curve $\mathcal{H}$ just described, projection onto configuration space yields a curve $\mathcal{G}$ — the curve traced by $\boldsymbol{x}(t)$—which is the opposite of unfamiliar, which can fairly be considered to mark the birthplace of modern physics; the elliptic figure of $\mathcal{G}$ was discovered observationally by Kepler, and Newton's accomplishment was to proceed from the figure to the underlying force law. Only after several decades did people—people of Jakob Hermann's generation—look to the reverse problem, the problem of deducing the orbit from the postulated force law. It is by entrenched tradition (but also for good reason: ask the founding fathers of theoretical mechanics) that we look upon

$$\text{force law} \quad \longrightarrow \quad \text{orbit}$$

as an allusion to the "direct" problem of dynamics, and it is to an elegantly swift approach to the solution of the "direct Kepler problem"—devised by Gibbs in the 1890's—that I now turn.[64]

The first part of Gibbs' masterful argument (to which I will soon return) culminates in the "invention" of $\boldsymbol{K}$ as a "constant of integration" (and a constant therefore of the motion).[65] We, however, are in position to consider that point already established, and to proceed directly to the final lines of his argument. Construct

$$\begin{aligned}\boldsymbol{x}\cdot\boldsymbol{K} &= \tfrac{1}{m}\,\boldsymbol{x}\cdot(\boldsymbol{p}\times\boldsymbol{L}) - \tfrac{k}{r}\,\boldsymbol{x}\cdot\boldsymbol{x}\\ &= \tfrac{1}{m}\,\boldsymbol{L}\cdot(\boldsymbol{x}\times\boldsymbol{p}) - kr\\ &= \tfrac{1}{m}\ell^2 - kr\end{aligned}$$

At (189) we established $\boldsymbol{K} = ke\hat{\boldsymbol{K}}$, so writing $\boldsymbol{x}\cdot\boldsymbol{K} = ker\cos\theta$ we have

$$r = \tfrac{\ell^2}{mk}\tfrac{1}{1+e\cos\theta} \tag{191}$$

which we recognize to be the polar description of an ellipse $\mathcal{G}$, with

- one focus at the origin;
- eccentricity e;
- principal axis pointed out by $\hat{\boldsymbol{K}}$.

I look back now to the first part of Gibbs' argument. From the equation of motion

$$m\ddot{\boldsymbol{x}} = -\tfrac{k}{r^3}\,\boldsymbol{x}$$

it follows that $\frac{d}{dt}(\boldsymbol{x}\times m\dot{\boldsymbol{x}}) = \boldsymbol{0}$, from which Gibbs obtains the angular momentum vector as a "constant of integration:"

$$\boldsymbol{x}\times m\dot{\boldsymbol{x}} = \boldsymbol{L} \quad : \quad \text{constant}$$

[64] My source is §61 Example 3 in J. W. Gibbs & E. B. Wilson's *Vector Analysis* (1901). Gibbs' primary intent was to demonstrate the utility of his new "vector analysis," and it was as an incidental by-product of his argument that he was led to reinvention of the $\boldsymbol{K}$-vector. As I understand the situation, is was from Gibbs that Runge borrowed his (similarly pedagogical) discussion, and from Runge that Lenz borrowed the $\boldsymbol{K}$ which he introduced into the "old quantum theory of the hydrogen atom," with results which are remembered only because they engaged the imagination of the young Pauli.[58]

[65] The argument was intended to illustrate the main point of Gibbs' §61, which is that "the... *integration* of vector equations in which the differentials depend upon scalar variables needs but a word."

Construct

$$\ddot{\boldsymbol{x}} \times \boldsymbol{L} = -\tfrac{k}{r^3}\,\boldsymbol{x} \times \boldsymbol{L}$$

and notice that

$$\begin{aligned}
\text{left hand side} &= \tfrac{d}{dt}\big\{m\dot{\boldsymbol{x}} \times \boldsymbol{L}\big\}\\
\text{right hand side} &= -\tfrac{mk}{r^3}\,\boldsymbol{x} \times (\boldsymbol{x} \times \dot{\boldsymbol{x}})\\
&= -\tfrac{mk}{r^3}\big\{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{x})\dot{\boldsymbol{x}}\big\}\\
&= -\tfrac{mk}{r^3}\big\{(r\dot{r})\boldsymbol{x} - r^2\dot{\boldsymbol{x}}\big\}\\
&= \tfrac{d}{dt}\big\{mk\tfrac{1}{r}\boldsymbol{x}\big\}
\end{aligned}$$

which entail

$$\dot{\boldsymbol{x}} \times \boldsymbol{L} = k\tfrac{1}{r}\boldsymbol{x} + \boldsymbol{K}$$

where

$$\boldsymbol{K} = \dot{\boldsymbol{x}} \times \boldsymbol{L} - k\tfrac{1}{r}\boldsymbol{x} \quad : \quad \text{constant of integration}$$

precisely reproduces the definition (188) of the "Runge-Lenz vector"! Gibbs makes no big deal of his accomplishment, cites no reference,[66] seems quite content to move directly to his immediate objective—the construction of $\mathcal{G}$.

In work cited earlier[58–60] one component of $\boldsymbol{K}$ emerges as a "separation constant," and the other components are obtained by "Poisson bracket closure." Gibbs' argument yields all components of $\boldsymbol{K}$ simultaneously, as a vector-valued "constant of integration"... which is horse of a different color.

The arguments used above to construct $\mathcal{G}$ and $\mathcal{H}$ are charming in their simple brevity, and both hinge on properties of $\boldsymbol{K}$. How does it happen that such arguments were much better known a century and more ago than they are today? The answer, I think, can be found in the onset of a pedagogical tradition which treats the Kepler problem as an incidental instance of the *general* central force problem—a tradition which de-emphasizes all that is *special about and peculiar to* the Kepler problem (most notably, the existence of the conserved vector $\boldsymbol{K}$). The material thus eliminated is, of course, precisely the material most directly relevant to understanding "multiple separability," "hidden symmetry" and other deep ramifications of Bertrand's theorem.

One dangling detail: It has been asserted that "$\boldsymbol{K}$ is directed toward the perihelion." We are in position now to supply the demonstration. At perihelion, (188) assumes the form

$$\begin{aligned}
\boldsymbol{K} = {}&(\text{vector of length } \ell^2/mr_{\text{min}} \text{ directed toward perihelion})\\
&+ (\text{vector of length } k \text{ directed toward aphelion})
\end{aligned}$$

---

[66] *Vector Analysis* was written by Wilson (Gibbs' recent student) on the basis of class notes Gibbs had been using for nearly twenty years. It contains no bibliography, no reference to the literature apart from an allusion to work of Heaviside and Föppl which can be found in Wilson's General Preface.

But it is an implication of (191) that

$$(\ell^2/mr_{\min}) - k = ke$$

so for non-circular orbits ($e > 0$) the former vector predominates, which secures the assertion. Note that in the contrary circumstance the sign of $\boldsymbol{K}_\perp \equiv \boldsymbol{L} \times \boldsymbol{K}$ would be reversed, the center of the hodograph would be displaced in the wrong direction, and the physical consequences (see again Figure 10) would be absurd. From results now in hand it becomes possible to state that the *center* of the orbital ellipse resides at

$$\boldsymbol{C} = -f\hat{\boldsymbol{K}} = -(f/ke)\boldsymbol{K} = -(a/k)\boldsymbol{K}$$

15. Stokes parameters for Keplerean orbits. The ellipses latent in light beams and contemplated by Stokes are drawn in "electric 2-space" and detected by photometric techniques special to optics, but are in all other respects—including the temporal aspects of their production—homologous to those traced in physical 2-space by harmonically bound mass points. Bound orbits encountered in connection with the Kepler problem are similarly figured ("an ellipse is an ellipse") and live similarly in "physical 2-space," but they are (with respect to the force center: see Figure 11) *eccentrically positioned*, and their *temporal production is distinctive*.[67] My objective here will be to develop the details of a point so small as to be almost obvious: Stoke's parameters can be pressed into service as "descriptors of Keplerean orbits" if the circumstances just italicized are discarded. The discussion will merit the effort if and to the extent that it opens a window onto the underlying *dynamics* of the Kepler problem.

Abandonment of "eccentric position" as a consideration makes it natural to place the coordinate origin at the *center* of the orbital ellipse. Such a step may seem alien to the physics of the problem, but is sanctioned by a tradition which is in fact older than the physics: Kepler himself—whose (pre-Newtonian!) work sought only to sift kinematical principles from the observational data—found it convenient (having established in his 1st Law that planetary orbits *are* elliptical) to work in the *principal axis frame of the orbital ellipse* (Figure 12). And so also will we: see Figure 13, from which I take my notation.

Keplerean ellipses are not generated by a "Lissajous process." Terms like "amplitude" and (especially) "relative phase" are therefore alien to the description of such curves. But the parameters $\{\mathfrak{X}_1, \mathfrak{X}_2\}$ which set the size of the "principal bounding box" are amplitude-like, and easily evaluated: by implicit differentiation of $ux^2 + 2wxy + vy^2 = 1$ we obtain

$$(ux + wy)\tfrac{dx}{dy} + (wx + vy) = 0$$

[67] This fact becomes vividly evident when one compares the Keplerean hodograph (found by Hamilton to be circular) with the that of an isotropic oscilator, which is shown in Figure 8 to be elliptical.

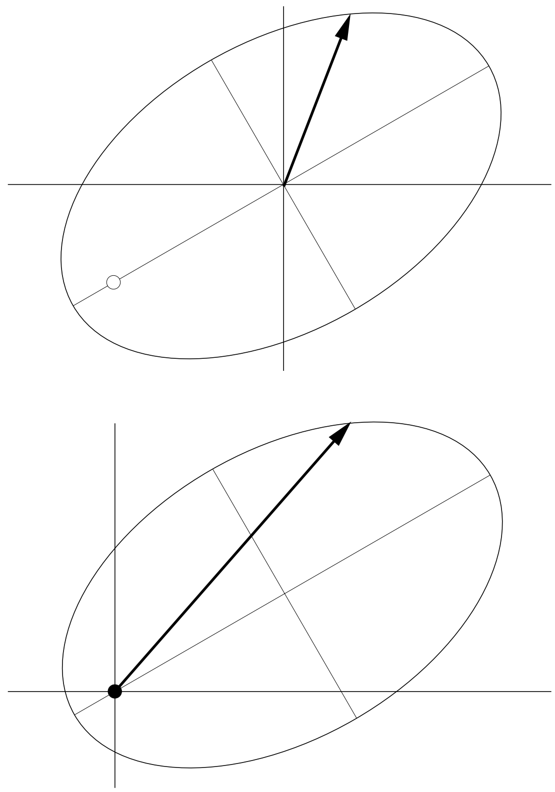

FIGURE 11: *Comparison of the ellipses that arise from the isotropic oscillator problem with those that arise from the Kepler problem. The former system (but not the latter) is—both geometrically and kinematically—homologous to the monochromatic beam problem contemplated by Stokes.*

From $\frac{dx}{dy} = 0$ we are led therefore to $y = -wx/v$ which, when introduced back into the equation which marked our point of departure gives $(uv-w^2)x^2/v = 1$. Thus are we led to equations

$$\mathcal{X}_1^2 = \frac{v}{uv - w^2}$$
$$\mathcal{X}_2^2 = \frac{u}{uv - w^2}$$

from which (13)

$$d^2 = \mathcal{X}_1^2 + \mathcal{X}_1^2 = \frac{u + v}{uv - w^2}$$

can be recovered as a corollary.

It is a remarkable property of Keplerean ellipses—and a property not shared by oscillator ellipses—that the semi-major axis is ℓ-independent, fixed

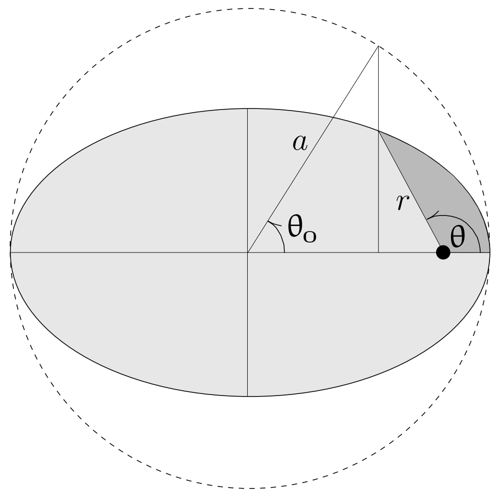

FIGURE 12: *Kepler's approach to the kinematic aspect of the orbital problem. The angle θ is in celestial mechanics called the "true anomaly." Kepler introduces an "auxiliary circle" which permits him to define also an "eccentric anomaly" θ_0. One has*[68]

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = a(1-\cos\theta_0)$$

giving

$$\tan\tfrac{1}{2}\theta = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}\theta_0 \qquad (i)$$

Working from his 2^{nd} *Law*

$$\frac{t}{T} = \frac{\text{area of heavily shaded sector}}{\text{total area of ellipse}}$$

(here t denotes elapsed time since perihelion, *and T is the* period: *according to the* 3^{rd} *Law $T \sim a^{\frac{3}{2}}$) Kepler obtains what has come to be called "**Kepler's equation**"*

$$\tau = \theta_0 - e\sin\theta_0 \quad \textit{with} \quad \tau \equiv 2\pi(t/T) \qquad (ii)$$

By functional inversion he would have $\theta_0(t)$, which on introduction into (i) gives $\theta(t)$. Over the years, literally hundreds of solutions of the functional inversion problem have been proposed, many of which are described in P. Colwell in his wonderful little book Solving Kepler's Problem over Three Centuries (1993)[69]. *We discover, for example, that it was Bessel's Fourier-analytic approach (1817) to the inversion of (ii) which led to the invention and development of the theory of Bessel functions.*

[68] Recall (191) and see *Relativistic Classical Fields* (1973), pp. 253–261.
[69] See also the end of Goldstein's §3-8.

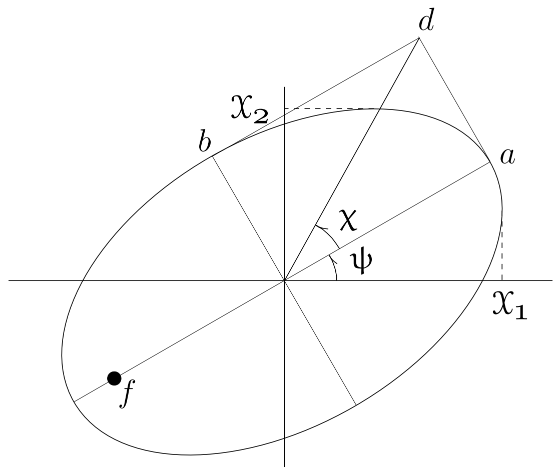

FIGURE 13: *Notation employed in description of Keplerean orbits. Present conventions are by design consistent with those employed in §1. Compare Figure 3.*

solely and entirely by the energy: $a = -k/2E$. On the other hand, the semi-minor axis b depends conjointly upon E and ℓ (and so also, therefore, do e, f and K): $b = \sqrt{-\ell^2/2mE}$. From $0 \leqslant b^2 \leqslant a^2$ we discover that E sets a bound on the possible values of ℓ:

$$0 \leqslant \ell^2 \leqslant -mk^2/2E \equiv \ell^2_{\text{max}} \tag{192}$$

The upshot of the preceding remarks is that E and ℓ serve to set the figure of the Keplerean, but one needs $\hat{\boldsymbol{K}}$ to fix its orientation.

A convenient measure of the "size" of a Keplerean orbit is provided by

$$\begin{aligned} d^2 &= a^2 + b^2 = \mathcal{X}_1^2 + \mathcal{X}_2^2 \quad : \quad \text{see again Figure 2} \\ &= \tfrac{k^2}{4E^2} - \tfrac{\ell^2}{2mE} = \begin{cases} a^2 & \text{if } \ell^2 = 0 \quad : \text{radial orbit} \\ 2a^2 & \text{if } \ell^2 = \ell^2_{\text{max}} : \text{circular orbit} \end{cases} \\ &= \tfrac{k^2}{4E^2}\big[1 + (\ell/\ell_{\text{max}})^2\big] \end{aligned} \tag{193}$$

Alternative measures of the "shape" of such an orbit are provided by the "eccentricity"

$$\begin{aligned} e = \sqrt{1-(b/a)^2} &= \sqrt{1+2E\ell^2/mk^2} = \sqrt{1-(\ell/\ell_{\text{max}})^2} \\ &= K/k \end{aligned} \tag{194}$$

and by

$$\begin{aligned} \tan\chi = b/a &= \sqrt{-2E\ell^2/mk^2} = \ell/\ell_{\text{max}} \\ &= \sqrt{1-e^2} \end{aligned} \tag{195}$$

The displacement of the force center from the orbital (geometric) center is given by

$$f = ea = -\sqrt{1 + 2E\ell^2/mk^2}\cdot(k/2E) = -K/2E \tag{196}$$

and the direction of that displacement ("orientation" of the orbital ellipse) is given by

$$\tan\psi = K_2/K_1 \tag{197}$$

Drawing upon these elementary identities

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} \quad\text{and}\quad \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

we obtain

$$\begin{aligned}
\cos 2\psi &= \frac{K_1^2 - K_2^2}{K_1^2 + K_2^2} \\
\sin 2\psi &= \frac{2K_1K_2}{K_1^2 + K_2^2} \\
\cos 2\chi &= \frac{\ell_{\text{max}}^2 - \ell^2}{\ell_{\text{max}}^2 + \ell^2} \quad:\ \text{notate}\ \frac{L_0^2 - L_3^2}{L_0^2 + L_3^2} \\
\sin 2\chi &= \frac{2\ell_{\text{max}}\ell}{\ell_{\text{max}}^2 + \ell^2} \quad:\ \text{notate}\ \frac{2L_0L_3}{L_0^2 + L_3^2}
\end{aligned}$$

Proceeding on the basis of (17), we are led to introduce "Keplerean Stokes parameters"

$$\left.\begin{aligned}
S_0 &\equiv d^2 \\
S_1 &\equiv S_0\cos 2\chi\cos 2\psi = d^2\,\frac{L_0^2 - L_3^2}{L_0^2 + L_3^2}\,\frac{K_1^2 - K_2^2}{K_1^2 + K_2^2} \\
S_2 &\equiv S_0\cos 2\chi\sin 2\psi = d^2\,\frac{L_0^2 - L_3^2}{L_0^2 + L_3^2}\,\frac{2K_1K_2}{K_1^2 + K_2^2} \\
S_3 &\equiv S_0\sin 2\chi \qquad\qquad\ = d^2\,\frac{2L_0L_3}{L_0^2 + L_3^2}
\end{aligned}\right\} \tag{198}$$

which are transparently redundant in the familiar ("monochromatic") sense

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = 0 \tag{199}$$

It becomes possible at this point to associate the Keplerean orbits which can be drawn on a given orbital plane—distinguished one from another by shape, orientation, helicity and size—with the points $\boldsymbol{S}$ in a 3-dimensional "mechanical Stokes space," and to associate orbits of given shape, orientation and helicity (size factored out) with points on the surface of a "mechanical Poincaré sphere." Helicity reversal is accomplished physically by $L_3 \to -L_3$ (i.e., by $\ell \to -\ell$), which by (198) becomes

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \to \begin{pmatrix} +S_0 \\ +S_1 \\ +S_2 \\ -S_3 \end{pmatrix} \quad:\quad \text{north/south hemisphere exchange}$$

We note in this connection that the parameters (198) are insensitive to

$$\boldsymbol{K} \to -\boldsymbol{K} \quad : \quad \text{force center/empty focus exchange}$$

It becomes possible to speak of "oppositely polarized" orbits, in exact imitation of optical practice. . . but to speak of the "non-interference of superimposed orbits" would be to speak physical celestial nonsense. It would be similarly empty of physical meaning to speak of a "relative phase," defined

$$\tan\delta \equiv S_3/S_2$$

in formal imitation of optical/oscillator realities.

While the parameters S_μ introduced into optics by Stokes bear the physical dimension of "intensity" (and to that circumstance owe much of their utility), the "mechanical Stokes parameters" introduced

- into oscillator mechanics at (159)
- into the mechanics of the Kepler problem (198)

are "squared lengths." In the former case one has $E \sim d^2$, and it became an easy matter to construct $\frac{1}{2}m\omega^2 S_\mu$ (dimension of energy, with S_0 = Hamiltonian) or $\frac{1}{2}(m\omega/\hbar)S_\mu$ (dimensionless). In the Keplerean case, on the other hand, one has this variant of (193)

$$E \sim -a^{-1} = -\frac{\sqrt{2-e^2}}{d} = -\frac{\sqrt{1+(\ell/\ell_{\max})^2}}{d}$$

and the relationship S_0 and energy (the Hamiltonian) becomes markedly more complicated. That complication touches on matters which I take up shortly, but first I would record this remark: from material native to the Kepler problem (m and k) is not possible to construct constants α and β such that

$$\begin{aligned} &\alpha\,(\text{length})^2 \quad \text{has the dimensions of energy}\\ &\beta\,(\text{length})^2 \quad \text{is dimensionless} \end{aligned}$$

but that importation of $\hbar$—though hardly natural to celestial interpretations of the Kepler problem!—makes it possible to achieve both objectives, for then one has access to the (generalized)

$$\text{Bohr radius} = \hbar^2/mk$$

giving $\beta = 1/(\text{Bohr radius})^2$ and $\alpha = k/(\text{Bohr radius})^3$. Astronomers would find it more natural to introduce some "conveniently arbitrary" length, such as the "astronomical unit;" interconversion is accomplished (in the case $k = e^2$) by

$$\text{astronomical unit} = 2.828 \times 10^{21} \cdot \text{Bohr radius}$$

It was in direct and literal imitation of Stokes that were were led at (198) to introduce expressions $S_\mu(E, \ell, \hat{\boldsymbol{K}}\,)$ which, while they do serve as *descriptors* *of Keplerean orbits*, can hardly be argued to do their work more efficiently/informatively than the raw constants of motion upon which they depend. If (198) were the end of the story, it would not be a story worth telling. But equations (198) mark not the end of the story, but only its beginning.

At (184) we introduced certain conserved observables J_μ which were found by heavy calculation—therefore with a satisfying element of surprise—to satisfy

$$J_0^2 - J_1^2 - J_2^2 - J_3^2 = 0 \tag{186.3}$$

The expressions S_μ are, on the other hand, by nature not "observables" but number-valued functions of orbital elements, and no element of surprise attaches to the circumstance that they satisfy

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = 0 \tag{199}$$

which they do in consequence of this interesting but elementary identity:

$$\Big[\frac{x^2-y^2}{x^2+y^2}\Big]^2 + \Big[\frac{2xy}{x^2+y^2}\Big]^2 = \frac{|(x+iy)^2|}{(|x+iy|)^2} = 1 \quad : \quad \text{all } x \text{ and } y$$

Note that $\ell^2_{\max} \equiv L_0^2 = -\frac{1}{2}mk^2/E =$ numerical value of $\big\{-\frac{1}{2}mk^2/H \equiv J_0^2\big\}$, which in combination with (184) means that in J-notation (198) can be rendered

$$\left.\begin{aligned}
S_0 &\equiv d^2\\
S_1 &\equiv S_0\cos 2\chi\cos 2\psi = d^2\frac{J_0^2-J_3^2}{J_0^2+J_3^2}\frac{J_1^2-J_2^2}{J_1^2+J_2^2} = d^2\frac{J_1^2-J_2^2}{J_0^2+J_3^2}\\
S_2 &\equiv S_0\cos 2\chi\sin 2\psi = d^2\frac{J_0^2-J_3^2}{J_0^2+J_3^2}\frac{2J_1J_2}{J_1^2+J_2^2} = d^2\frac{2J_1J_2}{J_0^2+J_3^2}\\
S_3 &\equiv S_0\sin 2\chi \qquad\quad\;\; = d^2\frac{2J_0J_3}{J_0^2+J_3^2}
\end{aligned}\right\} \tag{200}$$

Use was made here of the "surprising" fact that $J_0^2 - J_3^2 = J_1^2 + J_2^2$, so at (200) that same element of surprise has been introduced into (199). Notice next that

$$\begin{aligned}
d^2 = \frac{k^2}{4E^2}\frac{L_0^2+L_3^2}{L_0^2} &= -\frac{1}{2mE}(L_0^2+L_3^2)\\
&= \text{numerical value of } \Big\{-\tfrac{1}{2mH}(J_0^2+J_3^2)\Big\}
\end{aligned}$$

and obtain

$$\left.\begin{aligned}
S_0 &= -\tfrac{1}{2mH}(J_0^2+J_3^2)\\
S_1 &= -\tfrac{1}{2mH}(J_1^2-J_2^2)\\
S_2 &= -\tfrac{1}{2mH}2J_1J_2\\
S_3 &= -\tfrac{1}{2mH}2J_0J_3
\end{aligned}\right\} \tag{201}$$

at which point the parameters S_μ have been promoted to the status of conserved *observables*, of which (198) refers to the *valuations* in specific cases. And (199) has become a corollary of (186.3): using the mark $\sim$ to signal surpression of a distracting $\left(\frac{1}{2mH}\right)^2$-factor, we have

$$\begin{aligned} S_1^2 + S_2^2 &\sim (J_1^2 + J_2^2)^2 = (J_0^2 - J_3^2)^2 \quad \text{by (186.3)} \\ \therefore\;\; S_1^2 + S_2^2 + S_3^2 &\sim (J_0^2 + J_3^2)^2 \sim S_0^2 \end{aligned}$$

Equations (201) are much easier to look at that than the equations (198) from which they descend, and to which they remain equivalent. It becomes worthwhile to see what (201) have to say when evaluated; one finds

$$\left.\begin{aligned} S_0 &= \frac{k^2}{4E^2} - \frac{\ell^2}{2mE} &&= d^2 \\ S_1 &= \frac{1}{4E^2}(K_1^2 - K_2^2) &&= d^2 \cos 2\chi \cos 2\psi \\ S_2 &= \frac{1}{4E^2} 2K_1K_2 &&= d^2 \cos 2\chi \sin 2\psi \\ S_3 &= \sqrt{-\frac{k^2}{2mE^3}}\,\ell &&= d^2 \sin 2\chi \end{aligned}\right\} \tag{202}$$

from which (199) follows as a consequence of (187); i.e., of

$$K^2 \equiv K_1^2 + K_2^2 = k^2 + \tfrac{2}{m}E\ell^2$$

The geometrical parameters identified in Figure 13 can therefore be described

$$d^2 = S_0 \tag{203.1}$$

$$\cos 2\chi = S_3\Big/\sqrt{S_1^2 + S_2^2} \;\;\Rightarrow\;\; \begin{cases} \cos^2\chi = \frac{1}{2}\big[S_0 + \sqrt{S_1^2 + S_2^2}\,\big]/S_0 \\ \sin^2\chi = \frac{1}{2}\big[S_0 - \sqrt{S_1^2 + S_2^2}\,\big]/S_0 \end{cases} \tag{203.2}$$

$$a^2 = \tfrac{1}{2}\big[S_0 + \sqrt{S_1^2 + S_2^2}\,\big] \tag{203.3}$$

$$b^2 = \tfrac{1}{2}\big[S_0 - \sqrt{S_1^2 + S_2^2}\,\big] \tag{203.4}$$

$$f^2 = \sqrt{S_1^2 + S_2^2} \tag{203.5}$$

$$\tan 2\psi = S_2/S_1 \;\;\Rightarrow\;\; \begin{cases} \cos^2\psi = \frac{1}{2}\big[1 + S_1\big/\sqrt{S_1^2 + S_2^2}\,\big] \\ \sin^2\psi = \frac{1}{2}\big[1 - S_1\big/\sqrt{S_1^2 + S_2^2}\,\big] \end{cases} \tag{203.6}$$

One has also these descriptions of the physical parameters:

$$E = -\tfrac{k}{2}\Big[\tfrac{1}{2}\big[S_0 + \sqrt{S_1^2 + S_2^2}\,\big]\Big]^{-\frac{1}{2}} \tag{204.1}$$

$$\ell = \tfrac{1}{2}\sqrt{mk}\,S_3\Big[\tfrac{1}{2}\big[S_0 + \sqrt{S_1^2 + S_2^2}\,\big]\Big]^{-\frac{3}{4}} \tag{204.2}$$

$$K = ke = k\Big[2\sqrt{S_1^2 + S_2^2}\Big/\big[S_0 + \sqrt{S_1^2 + S_2^2}\,\big]\Big]^{\frac{1}{2}} \tag{204.3}$$

One can use (199) to cast the right sides of the preceding formulae in a great variety of alternative forms. Equations (203) bear no special relationship to the Kepler problem; they pertain to the geometry of ellipses-in-general, and might have been stated in §1. Equations (204), on the other hand, are special to Keplerean orbits.

16. A deeper look: parabolic coordinates again. It is a lesson of experience that to look with fresh depth into Kepler problem one should put on confocal parabolic eyeglasses.[70] Write x for x_1, y for x_2, introduce parabolic coordinates $\{\mu,\nu\}$ by means of equations[71]

$$\left.\begin{aligned} x &= \tfrac{1}{2r}(\mu^2-\nu^2)\\ y &= \tfrac{1}{r}\mu\nu \end{aligned}\right\} \tag{205}$$

and construct the associated momenta by covariant vector transformation

$$\begin{aligned} p_\mu &= \tfrac{\partial x}{\partial\mu}p_x + \tfrac{\partial y}{\partial\mu}p_y = \tfrac{1}{r}(+\mu p_x + \nu p_y)\\ p_\nu &= \tfrac{\partial x}{\partial\nu}p_x + \tfrac{\partial y}{\partial\nu}p_y = \tfrac{1}{r}(-\nu p_x + \mu p_y)\\ &\Downarrow\\ p_x &= \tfrac{r}{\mu^2+\nu^2}(\mu p_\mu - \nu p_\nu)\\ p_y &= \tfrac{r}{\mu^2+\nu^2}(\nu p_\mu + \mu p_\nu) \end{aligned}$$

The transformation $\{x,y,p_x,p_y\} \longleftarrow \{\mu,\nu,p_\mu,p_\nu\}$ provides an instance of a so-called "extended point transformation,"[72] and its canonicity is therefore assured. Parabolic coordinates are recommended to our Keplerean attention by two circumstances:

$$\begin{aligned} x^2+y^2 &= \tfrac{1}{4r^2}(\mu^2+\nu^2)^2\\ p_x^2+p_y^2 &= \tfrac{r^2}{\mu^2+\nu^2}(p_\mu^2+p_\nu^2) \end{aligned}$$

It follows that the Keplerean Hamiltonian can be expressed

$$H = \tfrac{r^2}{\mu^2+\nu^2}\left\{\tfrac{1}{2m}(p_\mu^2+p_\nu^2) - \tfrac{2k}{r}\right\} \tag{206}$$

Dynamical curves $\mathcal{C}$ are inscribed on isoenergetic surfaces Σ_E within the 4-dimensional $\{\mu,\nu,p_\mu,p_\nu\}$-coordinatized phase space of the problem. For

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[70] The origin of this insight appears to be lost in dim antiquity. It was already ancient when imported into the old quantum theory of the hydrogen atom (Stark effect), and is implicit in work (on the "two centers problem") done by Euler in 1760. I suspect one must look to someone of Jacobi's generation to find the first explicit application of parabolic coordinates to study of the Kepler problem; see, in this connection, pp. 261–264 in H. C. Corben & P. Stehle, *Classical Mechanics* (1950).

[71] Here r is a "length," of arbitary value—carefully *not* to be confused with $\sqrt{x^2+y^2}$. For related material, see p. 30 in "Reduced Kepler problem in elliptic coordinates" (1999).

[72] I take my terminology from E. T. Whittaker, *Analytical Dynamics* (1937), p. 293.

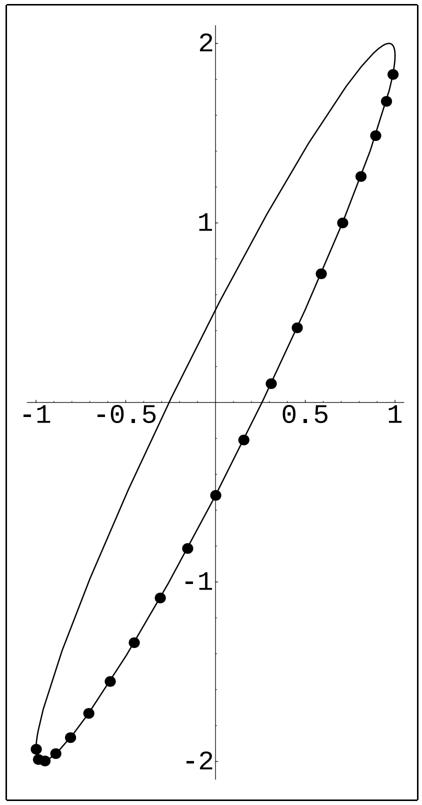

FIGURE 14: *Isotropic oscillator motion on the* $\{\mu,\nu\}$*-plane, with ticks marking progress through half a cycle.*

bound orbits $E < 0$, which we might emphasize by writing $E = -\frac{1}{2}mr^2\omega^2$. We then have

$$\tfrac{1}{2m}(p_\mu^2 + p_\nu^2) + \tfrac{1}{2}m\omega^2(\mu^2 + \nu^2) = 2k/r \tag{207}$$

which resembles an equation

$$\tfrac{1}{2m}(p_x^2 + p_y^2) + \tfrac{1}{2}m\omega^2(x^2 + y^2) = E$$

we might write if we had oscillators once again on our minds! The Kepler problem is from this point of view *equivalent to a* ω*-parameterized population of isotropic oscillator problems*, in each of which we have interest only in solutions of "energy" $2k/r$.

The general solution of (207) can be described

$$\left.\begin{aligned} \mu(t) &= \mathcal{F}_1\cos(\omega t) \\ \nu(t) &= \mathcal{F}_2\cos(\omega t + \theta) \end{aligned}\right\} \tag{208}$$

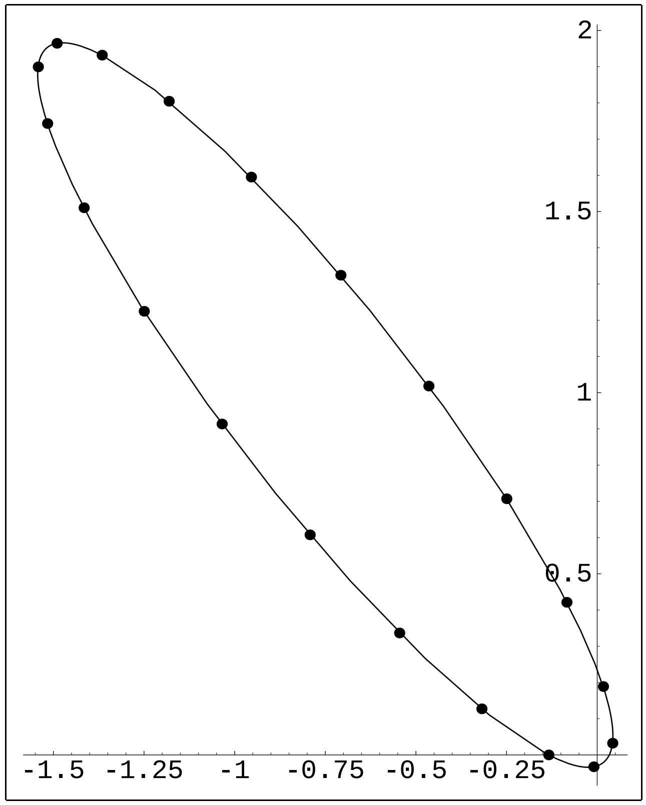

FIGURE 15: *(205) has been used to transcibe the preceding figure onto the* $\{x, y\}$*-plane. The formerly centered ellipse has become an ellipse with Keplerean placement. The ticks which formerly marked half a cycle are now distributed over one complete tour of the orbit. That is because* (μ, ν) *and* $(-\mu, -\nu)$ *are sent by (205) into the same point* (x, y)*; tours of* $\{\mu, \nu\}$*-ellipse become* duplicated *tours of its image. The ticks clearly do not conform to Kepler's Law of Areas, for reasons discussed in the text.*

δ are arbitrary, but $\omega \equiv \sqrt{-2E/mr^2}$ and $\mathcal{F} \equiv (\mathcal{F}_1^2 + \mathcal{F}_2^2)^{\frac{1}{2}}$ are constrained to satisfy $\omega\mathcal{F} = \sqrt{4k/mr}$, the practical effect of which is this: the prescribed value of $E < 0$ sets

$$\omega = \sqrt{-2E/mr^2} \quad \text{and} \quad \mathcal{F} = \sqrt{-2kr/E} \tag{209}$$

Plugging (208) into (205), we use elementary identities to obtain

$$\begin{aligned}
x(t) &= \tfrac{1}{4r}\big\{\mathcal{F}_1^2\big(1 + \cos(2\omega t)\big) - \mathcal{F}_2^2\big(1 + \cos(2\omega t + 2\theta)\big)\big\} \\
y(t) &= \tfrac{1}{2r}\mathcal{F}_1\mathcal{F}_2\big\{\cos(\theta) + \cos(2\omega t + \theta)\big\}
\end{aligned}$$

which can be expressed

$$\left.\begin{aligned} x(t) &= x_0 + \mathcal{X}_1 \cos(2\omega t + \delta_1) \\ y(t) &= y_0 + \mathcal{X}_2 \cos(2\omega t + \delta_2) \end{aligned}\right\} \tag{210}$$

with

$$\left.\begin{aligned} x_0 &= \tfrac{1}{4r}\big\{\mathcal{F}_1^2 - \mathcal{F}_2^2\big\} \\ \mathcal{X}_1 &= \tfrac{1}{4r}\sqrt{\mathcal{F}_1^4 + \mathcal{F}_2^4 - 2\mathcal{F}_1^2\mathcal{F}_2^2\cos 2\theta} \\ \delta_1 &= \arctan\left\{\frac{\mathcal{F}_2^2 \sin 2\theta}{\mathcal{F}_2^2\cos 2\theta - \mathcal{F}_1^2}\right\} \\ y_0 &= \tfrac{1}{4r}2\mathcal{F}_1\mathcal{F}_2\cos\theta \\ \mathcal{X}_2 &= \tfrac{1}{4r}2\mathcal{F}_1\mathcal{F}_2 \\ \delta_2 &= \theta \end{aligned}\right\} \tag{211.1}$$

If we write $\mathcal{F}_1 = \mathcal{F}\cos\varphi$, $\mathcal{F}_2 = \mathcal{F}\sin\varphi$ and consider $\mathcal{F}$ to be set by the energy but φ to be adjustable, then we have

$$\left.\begin{aligned} x_0 &= \tfrac{1}{4r}\mathcal{F}^2\cos 2\varphi \\ \mathcal{X}_1 &= \tfrac{1}{4r}\mathcal{F}^2\sqrt{1 - \sin^2 2\varphi\cdot\cos^2\theta} \\ \delta_1 &= \arctan\left\{\frac{\sin^2\varphi\cdot\sin 2\theta}{\sin^2\varphi\cdot\cos 2\theta - \cos^2\varphi}\right\} \\ y_0 &= \tfrac{1}{4r}\mathcal{F}^2\sin 2\varphi\cdot\cos\theta \\ \mathcal{X}_2 &= \tfrac{1}{4r}\mathcal{F}^2\sin 2\varphi \\ \delta_2 &= \theta \end{aligned}\right\} \tag{211.2}$$

If, in particular, we set $\mathcal{F}_1^2 = \mathcal{F}_2^2 = \frac{1}{2}\mathcal{F}^2$ (i.e., if we set $\varphi = 45°$) and $\theta = -90°$ then $\mu(t) = \frac{1}{\sqrt{2}}\mathcal{F}\cos\omega t$, $\nu(t) = \frac{1}{\sqrt{2}}\mathcal{F}\cos(\omega t - \frac{1}{2}\pi) = \frac{1}{\sqrt{2}}\mathcal{F}\sin\omega t$ trace on the $\{\mu,\nu\}$-plane a centered circle of radius $\frac{1}{\sqrt{2}}\mathcal{F}$ while, whether we work from (211.1) or (211.2), we have $x_0 = y_0 = 0$, $\mathcal{X}_1 = \mathcal{X}_2 = \frac{1}{4r}\mathcal{F}^2$, $\delta_1 = 0$ and $\delta_2 = -90°$; (210) therefore read $x(t) = \frac{1}{4r}\mathcal{F}^2\cos 2\omega t$, $y(t) = \frac{1}{4r}\mathcal{F}^2\sin 2\omega t$, which trace (in duplicate) on the $\{x, y\}$-plane a centered circle of radius $\frac{1}{4r}\mathcal{F}^2$. The assertion that centered circles

$$\odot \text{ on the } \{x,y\}\text{-plane} \quad\longleftarrow\quad \odot \text{ on the } \{\mu,\nu\}\text{-plane}$$

was implicit in the identity $x^2 + y^2 = \frac{1}{4r^2}(\mu^2 + \nu^2)^2$ encountered earlier, and its recovery can be read as a test of the accuracy of (211); it is a special case of the more general assertion that (Figure 15)

$$\text{confocal ellipses} \quad\longleftarrow\quad \text{centered ellipses}$$

and it is to details of the latter that I now turn.

The center of the Cartesian ellipse lies at a distance f from the origin, with

$$\begin{aligned} f^2 &= x_0^2 + y_0^2 \\ &= \big(\tfrac{1}{4r}\big)^2\big\{(\mathcal{F}_2^2+\mathcal{F}_2^2)^2 - 4\mathcal{F}_1^2\mathcal{F}_2^2\sin^2\theta\big\} \\ &= \big(\tfrac{1}{4r}\big)^2\mathcal{F}^4\big\{1-\sin^2 2\varphi\cdot\sin^2\theta\big\} \end{aligned} \tag{212.1}$$

Taking the "size" of the ellipse to be given by the semi-diagonal d of every circumscribed rectangle (Figure 2), we have

$$\begin{aligned} d^2 &= \mathcal{X}_1^2 + \mathcal{X}_2^2 \\ &= \big(\tfrac{1}{4r}\big)^2\big\{(\mathcal{F}_1^2+\mathcal{F}_2^2)^2 + 4\mathcal{F}_1^2\mathcal{F}_2^2\sin^2\theta\big\} \\ &= \big(\tfrac{1}{4r}\big)^2\mathcal{F}^4\big\{1+\sin^2 2\varphi\cdot\sin^2\theta\big\} \end{aligned} \tag{212.2}$$

The slope of the principal axis is given by[73]

$$\begin{aligned} \tan\psi &= y_0/x_0 \\ &= \tan 2\varphi\cdot\cos\theta \end{aligned} \tag{212.3}$$

Look to the case $\cos\theta = 0$: the principal axis is then coincident with the x-axis, and we have

$$\begin{aligned} a &= \mathcal{X}_1 \\ &= \tfrac{1}{4r}\mathcal{F}^2 &&\text{in this special case} \\ b &= \mathcal{X}_2 \\ &= \tfrac{1}{4r}\mathcal{F}^2\sin 2\varphi \leqslant a &&\text{in this special case} \end{aligned}$$

giving

$$\begin{aligned} (\text{focal distance})^2 &= a^2 - b^2 \\ &= \big(\tfrac{1}{4r}\big)^2\mathcal{F}^4\big\{1-\sin^2 2\varphi\big\} \\ &= f^2 \quad \text{as evaluated in this special case} \end{aligned}$$

We will proceed in the assumption[74] that the lesson of the special case

$$f = \text{focal distance}$$

does in fact hold generally; i.e., that the ellipses here in question are *confocal at the Cartesian origin*. Working from $d^2 = a^2 + b^2$ and $f^2 = a^2 - b^2$ with the

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[73] It is to avoid the uninformative tedium of explicit proof that I am content here simply to *assume* that the principal axis passes in every case through the Cartesian origin…

[74] …and my intent here is similar; proof can be accomplished by the methods of §1, but the details are heavy and uninformative. I stand in need of a sharper mode of argument to establish these points which are, in themselves, almost obvious.

aid of (212.1&2) we then have

$$\begin{aligned} a^2 &= \tfrac{1}{2}(d^2 + f^2) \\ &= \left(\tfrac{1}{4r}\right)^2 \mathcal{F}^4 \qquad (212.4) \\ b^2 &= \tfrac{1}{2}(d^2 - f^2) \\ &= \left(\tfrac{1}{4r}\right)^2 \mathcal{F}^4 \sin^2 2\varphi \cdot \sin^2\theta \qquad (212.5) \end{aligned}$$

The shape of the ellipse is described by

$$\text{eccentricity} = \frac{f}{a} = \frac{2f}{d^2 + f^2}$$

but more conveniently by the angle χ (see again Figure 3), concerning which we have

$$\tan\chi = b/a = \sin 2\varphi \cdot \sin\theta \qquad (212.6)$$

Equations (212) provide a complete account of the elements of the confocal ellipse in terms of its centered precursor. Working from (20), we find that Stokes parameters descriptive of the latter can be described

$$\left.\begin{aligned} \Sigma_0 &= \mathcal{F}_1^2 + \mathcal{F}_2^2 &&= \mathcal{F}^2 \\ \Sigma_1 &= \mathcal{F}_1^2 - \mathcal{F}_2^2 &&= \mathcal{F}^2 \cos 2\varphi \\ \Sigma_2 &= 2\,\mathcal{F}_1\mathcal{F}_2\cos\theta &&= \mathcal{F}^2 \sin 2\varphi \cdot \cos\theta \\ \Sigma_3 &= 2\,\mathcal{F}_1\mathcal{F}_2\sin\theta &&= \mathcal{F}^2 \sin 2\varphi \cdot \sin\theta \end{aligned}\right\} \qquad (213)$$

while for the former we have

$$\begin{aligned} S_0 &= \mathcal{X}_1^2 + \mathcal{X}_2^2 \\ S_1 &= S_0 \cos 2\chi \cos 2\psi = S_0 \frac{1 - \tan^2\chi}{1 + \tan^2\chi}\,\frac{1 - \tan^2\psi}{1 + \tan^2\psi} \\ S_2 &= S_0 \cos 2\chi \sin 2\psi = S_0 \frac{1 - \tan^2\chi}{1 + \tan^2\chi}\,\frac{2\tan\psi}{1 + \tan^2\psi} \\ S_3 &= S_0 \sin 2\chi \qquad\quad\;\; = S_0 \frac{2\tan\chi}{1 + \tan^2\chi} \end{aligned}$$

Drawing upon (212) and (213) we have

$$S_0 = \left(\tfrac{1}{4r}\right)^2 \mathcal{F}^4 \left\{1 + \sin^2 2\varphi \cdot \sin^2\theta\right\} = \left(\tfrac{1}{4r}\right)^2 \left\{\Sigma_0^2 + \Sigma_3^2\right\}$$

Moreover

$$\begin{aligned} \tan^2\chi &= \;\sin^2 2\varphi \cdot \sin^2\theta = \Sigma_3^2/\Sigma_0^2 \\ \tan^2\psi &= \tan^2 2\varphi \cdot \cos^2\theta = \Sigma_2^2/\Sigma_1^2 \end{aligned}$$

so we have

$$\left.\begin{aligned}
S_0 &= \left(\tfrac{1}{4r}\right)^2(\Sigma_0^2+\Sigma_3^2)\\
S_1 &= \left(\tfrac{1}{4r}\right)^2(\Sigma_0^2-\Sigma_3^2)\frac{\Sigma_1^2-\Sigma_2^2}{\Sigma_1^2+\Sigma_2^2}\\
S_2 &= \left(\tfrac{1}{4r}\right)^2(\Sigma_0^2-\Sigma_3^2)\frac{2\Sigma_1\Sigma_2}{\Sigma_1^2+\Sigma_2^2}\\
S_3 &= \left(\tfrac{1}{4r}\right)^2 2\Sigma_0\Sigma_3
\end{aligned}\right\}\tag{214.1}$$

which describe the Stokes parameters of the Keplerean ellipse in terms of those of its harmonic precursor. We notice that (214.1) sends

- circular orbits $\mapsto$ circular orbits ($S_1=S_2=0 \Leftarrow \Sigma_1=\Sigma_2=0$);
- linear orbits $\mapsto$ linear orbits ($S_3=0\Leftarrow\Sigma_3=0$).

We notice also that $\Sigma_0^2-\Sigma_3^2=\Sigma_1^2+\Sigma_2^2$ permits further simplification:

$$\left.\begin{aligned}
S_0 &= \left(\tfrac{1}{4r}\right)^2(\Sigma_0^2+\Sigma_3^2)\\
S_1 &= \left(\tfrac{1}{4r}\right)^2(\Sigma_1^2-\Sigma_2^2)\\
S_2 &= \left(\tfrac{1}{4r}\right)^2 2\Sigma_1\Sigma_2\\
S_3 &= \left(\tfrac{1}{4r}\right)^2 2\Sigma_0\Sigma_3
\end{aligned}\right\}\tag{214.2}$$

And that from

$$\begin{aligned}
S_0^2-S_3^2 &= \left(\tfrac{1}{4r}\right)^4(\Sigma_0^4-2\Sigma_0^2\Sigma_3^2+\Sigma_3^4)\\
&= \left(\tfrac{1}{4r}\right)^4(\Sigma_0^2-\Sigma_3^2)^2\\
&= \left(\tfrac{1}{4r}\right)^4(\Sigma_1^2+\Sigma_2^2)^2\\
&= \left(\tfrac{1}{4r}\right)^4(\Sigma_1^4+2\Sigma_1^2\Sigma_2^2+\Sigma_1^4)\\
&= S_1^2+S_2^2
\end{aligned}$$

we discover it to be an implication of (214.2)—not at all surprising, yet gratifying—that

$$\Sigma_0^2-\Sigma_1^2-\Sigma_2^2-\Sigma_3^2=0 \quad\Rightarrow\quad S_0^2-S_1^2-S_2^2-S_3^2=0$$

Equations (214.2) display the parameters S_μ as quadratic combinations of their harmonic counterparts, and can be expressed

$$S_\mu = \left(\tfrac{1}{4r}\right)^4\boldsymbol{\Sigma}^{\mathsf{T}}\mathbb{M}_\mu\boldsymbol{\Sigma}$$

with

$$\mathbb{M}_0\equiv\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\quad
\mathbb{M}_1\equiv\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}$$

$$\mathbb{M}_2\equiv\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\quad
\mathbb{M}_3\equiv\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

What is surprising is that the matrices $\mathbb{M}_\mu$ are, so far as I have been able to determine, quite devoid of algebraic interest; they would arise if one introduced

$$\text{“red” Pauli matrices of the design }\begin{pmatrix}\bullet & 0 & 0 & \bullet\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \bullet & 0 & 0 & \bullet\end{pmatrix}$$

$$\text{“green” Pauli matrices of the design }\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \bullet & \bullet & 0\\ 0 & \bullet & \bullet & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

and then discarded half of each type, but I can think of no rationale for such a procedure.

Let (214.2) be notated

$$\begin{aligned} S_0 &= \tilde\Sigma_0^2 + \tilde\Sigma_3^2\\ S_1 &= \tilde\Sigma_1^2 - \tilde\Sigma_2^2\\ S_2 &= 2\tilde\Sigma_1\tilde\Sigma_2\\ S_3 &= 2\tilde\Sigma_0\tilde\Sigma_3 \end{aligned} \qquad \ldots\text{with } \tilde\Sigma_\mu \equiv \big(\tfrac{1}{4r}\big)\Sigma_\mu \tag{215}$$

and look to the algebraic inversion problem. Eliminating $\tilde\Sigma_3$ between the outer pair of equations, we obtain $4\tilde\Sigma_0^4 - 4S_0\tilde\Sigma_0^2 + S_3^2 = 0$ giving

$$\left.\begin{aligned} \tilde\Sigma_0^2 &= \tfrac{1}{2}\big\{S_0 + \sqrt{S_0^2 - S_3^2}\,\big\}\\ \tilde\Sigma_3^2 &= S_0 - \tilde\Sigma_0^2\\ &= \tfrac{1}{2}\big\{S_0 - \sqrt{S_0^2 - S_3^2}\,\big\} \end{aligned}\right\} \tag{216.1}$$

where sign ambiguity on the radical has been resolved in such a way as to ensure $\tilde\Sigma_0^2 \geqslant \tilde\Sigma_3^2$. Working similarly from the inner pair of equations, we obtain

$$\left.\begin{aligned} \tilde\Sigma_1^2 &= \tfrac{1}{2}\big\{\sqrt{S_1^2 + S_2^2} + S_1\big\}\\ \tilde\Sigma_2^2 &= \tilde\Sigma_1^2 - S_1\\ &= \tfrac{1}{2}\big\{\sqrt{S_1^2 + S_2^2} - S_1\big\} \end{aligned}\right\} \tag{216.2}$$

where the sign ambituity has been resolved so as to achieve

$$S_0^2 - S_3^2 = S_1^2 + S_2^2 \quad\Rightarrow\quad \tilde\Sigma_0^2 - \tilde\Sigma_1^2 - \tilde\Sigma_2^2 - \tilde\Sigma_3^2 = 0$$

Final sign ambiguities would attend removal of the squares from $\tilde\Sigma_1^2$, $\tilde\Sigma_2^2$ and $\tilde\Sigma_3^2$, but I am content to omit that fussy discussion.

We note that inversion of (205)—which we are motivated now to renotate

$$\begin{aligned} x &= \tilde\mu^2 - \tilde\nu^2\\ y &= 2\tilde\mu\tilde\nu \end{aligned}$$

—involves manipulations identical to those just sketched. We are brought to the remarkable conclusion that the transformation which sent centered ellipses to confocal ellipses *replicates itself in Stokes space*, where, however, it serves to describe the transformed *figures* of the ellipses.

We note also that (215) bears a striking formal similiarity to equations

$$\begin{aligned} Q_0 &= a_1^*a_1 + a_2^*a_2 \\ Q_1 &= a_1^*a_1 - a_2^*a_2 \\ Q_2 &= a_1^*a_2 + a_2^*a_1 \\ iQ_3 &= a_1^*a_2 - a_2^*a_1 \end{aligned}$$

which arose at (163) from the dynamics of isotropic oscillators, concerning which...

I would stress that the transformation which has been seen to carry oscillator orbits into Keplerean orbits (and *vice versa*) does not carry the temporal aspects of oscillator dynamics into those of Keplerean dynamics. This was made evident by the ticks in Figure 15, and arises from the circumstance that when we interpreted (207) as a Kepler-inspired invitation to do "oscillator physics" we

- demoted the Hamiltonian to the status of a constant $(\frac{1}{2}m\omega^2)$, and
- promoted a constant $(2k/r)^{71}$ to the status of a Hamiltonian.

The kinematic consequences of this adjustment become clear when one looks back again to Figure 12; the parabolic transform of harmonic oscillation can, in notation developed there, be rendered

$$\tau = \theta_0$$

while according to Kepler himself we should, for planetary motion, expect

$$\tau = \theta_0 - e\sin\theta_0$$

The point at issue can be phrased yet another way: if

- $\mathcal{O}_{\text{Kepler}} \equiv \mathcal{C}_{\text{Kepler}}$ projected onto configuration space
- $\mathcal{O}_{\text{oscillator}} \equiv \mathcal{C}_{\text{oscillator}}$ projected onto configuration space

then we have achieved

$$\mathcal{O}_{\text{Kepler}} \xleftrightarrow[\text{parabolic association}]{\hspace{4cm}} \mathcal{O}_{\text{oscillator}}$$

but the same cannot be said of $\mathcal{C}_{\text{Kepler}}$ and $\mathcal{C}_{\text{oscillator}}$, which yield distinct hodographs when projected onto momentum space.

17. "Mechanical Stokes parameters" as generators of canonical transformations. Stokes' parameters S_μ began life as optical "ellipse descriptors"—distinguished from others (and recommended) by the circumstance that they are accessible to direct measurement. In mechanical applications the latter (specifically optical) recommentation loses its force, but the "mechanical Stokes parameters" retain their value as ellipse descriptors *par excellance* and acquire, in addition, a new (specifically mechanical) claim to our attention. It is the latter—adumbrated near the end of §12—which I propose now to explore.

We were led at (165) to the identification of

$$\left.\begin{aligned} L_0(x_1,x_2,p_1,p_2) &= \tfrac{1}{4m\omega}(p_1^2+m^2\omega^2x_1^2)+\tfrac{1}{4m\omega}(p_2^2+m^2\omega^2x_2^2)\\ &= \tfrac{1}{2\omega}H_{\text{oscillator}}\\ L_1(x_1,x_2,p_1,p_2) &= \tfrac{1}{4m\omega}(p_1^2+m^2\omega^2x_1^2)-\tfrac{1}{4m\omega}(p_2^2+m^2\omega^2x_2^2)\\ L_2(x_1,x_2,p_1,p_2) &= \tfrac{1}{2m\omega}(p_1p_2+m^2\omega^2x_1x_2)\\ L_3(x_1,x_2,p_1,p_2) &= \tfrac{1}{2}(x_1p_2-x_2p_1) \end{aligned}\right\} \tag{217}$$

as a quartet of observables natural to the physics of isotropic oscillators, and at (185/6) to the identification of

$$\left.\begin{aligned} J_0(x_1,x_2,p_1,p_2) &= \sqrt{-\tfrac{1}{2}mk^2/H_{\text{Kepler}}}\\ &\qquad H_{\text{Kepler}} = \tfrac{1}{2m}(p_1^2+p_2^2)-k(x_1^2+x_2^2)^{-\frac{1}{2}}\\ J_1(x_1,x_2,p_1,p_2) &= \sqrt{-\tfrac{m}{2H}}\Big\{+\tfrac{1}{m}p_2J_3-kx_1(x_1^2+x_2^2)^{-\frac{1}{2}}\Big\}\\ J_2(x_1,x_2,p_1,p_2) &= \sqrt{-\tfrac{m}{2H}}\Big\{-\tfrac{1}{m}p_1J_3-kx_2(x_1^2+x_2^2)^{-\frac{1}{2}}\Big\}\\ J_3(x_1,x_2,p_1,p_2) &= x_1p_2-x_2p_1 \end{aligned}\right\} \tag{218}$$

similarly natural to the Kepler problem. Though these quartets differ radically in design, they share some essential features. To discuss those, we introduce a generic notation

$$\text{let } G_\mu \text{ signify} \begin{cases} L_\mu \text{ in the oscillatory case}\\ J_\mu \text{ in the Keplerean case}\end{cases}$$

and notice first off that G_μ bears the physical dimension of angular momentum or *action*. With the indispensable assistance of *Mathematica* one can easily establish—independently of all that has gone before—that

$$G_0^2-G_1^2-G_2^2-G_3^2=0 \tag{219}$$

and that

$$[G_i,G_j]=G_k \quad : \quad \big\{i,j,k\big\} \text{ cyclic on } \big\{1,2,3\big\} \tag{220}$$

from which it follows that (for example)

$$\begin{aligned} [G_1,G_0^2] &= 2G_0[G_1,G_0]\\ &= 2G_2[G_1,G_2]+2G_3[G_1,G_3]\\ &= 2(G_2G_3-G_3G_2)\\ &= 0 \end{aligned}$$

giving (as direct calculation would laboriously confirm)

$$[G_i, G_0] = 0 \quad : \quad i = 1, 2, 3$$

This last statement amounts to an assertion that the G_μ are (singly redundant) constants of the motion:

$$[H, G_\mu] = 0$$

Their constant numerical values can be read as a reference to the immobile figure of the orbital ellipse, that reference being direct[75]

$$S_\mu = \tfrac{4}{m\omega} L_\mu \tag{221}$$

in the oscillatory case, but indirect[76]

$$\left.\begin{aligned} S_0 &= \tfrac{1}{m^2k^2} J_0^2 (J_0^2 + J_3^2) \\ S_1 &= \tfrac{1}{m^2k^2} J_0^2 (J_1^2 - J_2^2) \\ S_2 &= \tfrac{1}{m^2k^2} J_0^2 (2J_1 J_2) \\ S_3 &= \tfrac{1}{m^2k^2} J_0^2 (2J_0 J_3) \end{aligned}\right\} \tag{222}$$

in the Keplerean case. We verify that in both cases the S_μ bear the physical dimension of $(\text{length})^2$. And in both cases one has

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = 0 \tag{223}$$

But the Poisson brackets $[S_\mu, S_\nu]_{\text{oscillator}}$ and $[S_\mu, S_\nu]_{\text{Kepler}}$ are readily shown to be distinct, even though those of L_μ precisely mimic those of J_μ; the discussion, carried beyond this point, resolves therefore into cases... for which the following general remarks are intended to be preparatory:

If $G(x_1, x_2, p_1, p_2)$ is an arbitrary observable and if the associated *Lie derivative operator* $\mathbf{D}_G$ is defined

$$\mathbf{D}_G \equiv [G, \bullet] = \frac{\partial G}{\partial x_1}\frac{\partial}{\partial p_1} + \frac{\partial G}{\partial x_2}\frac{\partial}{\partial p_2} - \frac{\partial G}{\partial p_1}\frac{\partial}{\partial x_1} - \frac{\partial G}{\partial p_2}\frac{\partial}{\partial x_2} \tag{224}$$

then

$$\begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} f_1\,(x_1, p_1, x_2, p_2; u) \\ g_1\,(x_1, p_1, x_2, p_2; u) \\ f_2\,(x_1, p_1, x_2, p_2; u) \\ g_2\,(x_1, p_1, x_2, p_2; u) \end{pmatrix} = e^{u\mathbf{D}_G} \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix} \tag{225}$$

describes the u-parameterized group of *canonical transformations generated* by the observable G, where the requirement that

$$[u][G] = \text{action}$$

[75] Use (163) in the final equation $S_\mu = (2\hbar/m\omega)Q_\mu$ of §12.

[76] Use (186.3) in (201).

From Jacobi's identity, written $[A,[B,X]]-[B,[A,X]]=[[A,B],X]$, we learn that

$$\mathbf{D}_A\mathbf{D}_B-\mathbf{D}_B\mathbf{D}_A=\mathbf{D}_{[A,B\,]} \tag{226}$$

which sets up a very pretty association of the form

$$\text{commutators} \longleftrightarrow \text{Poisson brackets}$$

In immediate consequence of (220) we therefore have

$$\left.\begin{aligned}
\mathbf{D}_{G_1}\mathbf{D}_{G_2}-\mathbf{D}_{G_2}\mathbf{D}_{G_1}&=\mathbf{D}_{G_3}\\
\mathbf{D}_{G_2}\mathbf{D}_{G_3}-\mathbf{D}_{G_3}\mathbf{D}_{G_2}&=\mathbf{D}_{G_1}\\
\mathbf{D}_{G_3}\mathbf{D}_{G_1}-\mathbf{D}_{G_1}\mathbf{D}_{G_3}&=\mathbf{D}_{G_2}
\end{aligned}\right\} \tag{227}$$

In short: the Lie operators associated with the observables G_1, G_2 and G_3 give rise to a commutator algebra which is identical to the Poisson bracket algebra satisfied by the G-observables themselves, and generate a *canonical representation* of the associated Lie group.

The transformation (225) lives, of course, in phase space, and when $[H,G]=0$ it serves to map dynamical flow lines onto dynamical flow lines. Such maps are, as Sophus Lie was the first to appreciate, most usefully studied in the infinitesimal limit:

$$\begin{pmatrix}x_1\\p_1\\x_2\\p_2\end{pmatrix}\longrightarrow\begin{pmatrix}x_1\\p_1\\x_2\\p_2\end{pmatrix}+\begin{pmatrix}\delta x_1\\\delta p_1\\\delta x_2\\\delta p_2\end{pmatrix}\quad\text{with}\quad\begin{pmatrix}\delta x_1\\\delta p_1\\\delta x_2\\\delta p_2\end{pmatrix}=\begin{pmatrix}\delta u\cdot[G,x_1]\\\delta u\cdot[G,p_1]\\\delta u\cdot[G,x_2]\\\delta u\cdot[G,p_2]\end{pmatrix} \tag{228.1}$$

The induced adjustment in the value of an arbitrary observable $A(x_1,p_1,x_2,p_2)$ becomes $A\longrightarrow A+\delta A$ with

$$\begin{aligned}
\delta A&=\frac{\partial A}{\partial x_1}\delta x_1+\frac{\partial A}{\partial p_1}\delta p_1+\frac{\partial A}{\partial x_2}\delta x_2+\frac{\partial A}{\partial p_2}\delta p_2\\
&=\delta u\cdot\left\{-\frac{\partial A}{\partial x_1}\frac{\partial G}{\partial p_1}+\frac{\partial A}{\partial p_1}\frac{\partial G}{\partial x_1}-\frac{\partial A}{\partial x_2}\frac{\partial G}{\partial p_2}+\frac{\partial A}{\partial p_2}\frac{\partial G}{\partial x_2}\right\}\\
&=\delta u\cdot[G,A]
\end{aligned} \tag{228.2}$$

from which (228.1) can be recovered as special cases

CANONICAL TRANSFORM THEORY IN THE OSCILLATORY CASE

Bringing (228) to (217) and drawing upon *Mathematica* for some computational assistance, we find more particularly that

$$\left.\begin{aligned}
\begin{pmatrix}\delta x_1\\ \delta p_1\\ \delta x_2\\ \delta p_2\end{pmatrix} &= \delta u_0\cdot\tfrac{1}{2}\begin{pmatrix}-\alpha^{-1}p_1\\ +\alpha^{+1}x_1\\ -\alpha^{-1}p_2\\ +\alpha^{+1}x_2\end{pmatrix} &&\text{if the generator is } L_0\\
\begin{pmatrix}\delta x_1\\ \delta p_1\\ \delta x_2\\ \delta p_2\end{pmatrix} &= \delta u_1\cdot\tfrac{1}{2}\begin{pmatrix}-\alpha^{-1}p_1\\ +\alpha^{+1}x_1\\ +\alpha^{-1}p_2\\ -\alpha^{+1}x_2\end{pmatrix} &&\text{if the generator is } L_1\\
\begin{pmatrix}\delta x_1\\ \delta p_1\\ \delta x_2\\ \delta p_2\end{pmatrix} &= \delta u_2\cdot\tfrac{1}{2}\begin{pmatrix}-\alpha^{-1}p_2\\ +\alpha^{+1}x_2\\ -\alpha^{-1}p_1\\ +\alpha^{+1}x_1\end{pmatrix} &&\text{if the generator is } L_2\\
\begin{pmatrix}\delta x_1\\ \delta p_1\\ \delta x_2\\ \delta p_2\end{pmatrix} &= \delta u_3\cdot\tfrac{1}{2}\begin{pmatrix}+\quad x_2\\ +\quad p_2\\ -\quad x_1\\ -\quad p_1\end{pmatrix} &&\text{if the generator is } L_3
\end{aligned}\right\}\tag{229}$$

where $\alpha \equiv m\omega$. Drawing upon $S_\mu \sim L_\mu$ and $[L_1, L_2] = L_3$, etc. we find that

$$\left.\begin{aligned}
L_0 \text{ induces } &\begin{cases} dS_0 = 0\\ dS_1 = 0\\ dS_2 = 0\\ dS_3 = 0\end{cases}\\
L_1 \text{ induces } &\begin{cases} dS_0 = 0\\ dS_1 = 0\\ dS_2 = +\delta u_1\cdot S_3\\ dS_3 = -\delta u_1\cdot S_2\end{cases}\\
L_2 \text{ induces } &\begin{cases} dS_0 = 0\\ dS_1 = -\delta u_2\cdot S_3\\ dS_2 = 0\\ dS_3 = +\delta u_2\cdot S_1\end{cases}\\
L_3 \text{ induces } &\begin{cases} dS_0 = 0\\ dS_1 = +\delta u_3\cdot S_2\\ dS_2 = -\delta u_3\cdot S_1\\ dS_3 = 0\end{cases}
\end{aligned}\right\}\tag{230}$$

To summarize: S_0, S_1, S_2 and S_3 are constant (which is to say: surfaces of constant S_0 ditto S_1 ditto S_2 ditto S_3 are invariant) with respect to the canonical transformations generated by L_0, and reciprocally: S_0 is constant with respect to the transformations generated by L_1, L_2 and L_3. With regard to the latter, the remainder of the story is told by the following equations:

$$\left.\begin{aligned}
\begin{pmatrix} dS_1 \\ dS_2 \\ dS_3 \end{pmatrix} &= \delta u_1 \cdot \mathbb{L}_1 \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \quad \text{with} \quad \mathbb{L}_1 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} dS_1 \\ dS_2 \\ dS_3 \end{pmatrix} &= \delta u_2 \cdot \mathbb{L}_2 \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \quad \text{with} \quad \mathbb{L}_2 \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} dS_1 \\ dS_2 \\ dS_3 \end{pmatrix} &= \delta u_3 \cdot \mathbb{L}_3 \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \quad \text{with} \quad \mathbb{L}_3 \equiv \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}\right\} \tag{231}$$

which describe rotations about the respective axes of the "mechanical Poincaré sphere."[77]

The result just so easily achieved can be displayed in a variety of other ways. Proceeding, for example, from (159), we have

$$\left.\begin{aligned}
dS_0 &= 2\mathfrak{X}_1 \cdot d\mathfrak{X}_1 + 2\mathfrak{X}_2 \cdot d\mathfrak{X}_2 \\
dS_1 &= 2\mathfrak{X}_1 \cdot d\mathfrak{X}_1 - 2\mathfrak{X}_2 \cdot d\mathfrak{X}_2 \\
dS_2 &= 2\mathfrak{X}_2 \cos\delta \cdot d\mathfrak{X}_1 + 2\mathfrak{X}_1 \cos\delta \cdot d\mathfrak{X}_2 - 2\mathfrak{X}_1\mathfrak{X}_2 \sin\delta \cdot d\delta \\
dS_3 &= 2\mathfrak{X}_2 \sin\delta \cdot d\mathfrak{X}_1 + 2\mathfrak{X}_1 \sin\delta \cdot d\mathfrak{X}_2 + 2\mathfrak{X}_1\mathfrak{X}_2 \cos\delta \cdot d\delta
\end{aligned}\right\} \tag{232}$$

Evidently the action of L_0 entails $d\mathfrak{X}_1 = d\mathfrak{X}_2 = d\delta = 0$. In the remaining cases we have

$$\begin{pmatrix} 2\mathfrak{X}_1 & -2\mathfrak{X}_2 & 0 \\ 2\mathfrak{X}_2 \cos\delta & 2\mathfrak{X}_1 \cos\delta & -2\mathfrak{X}_1\mathfrak{X}_2 \sin\delta \\ 2\mathfrak{X}_2 \sin\delta & 2\mathfrak{X}_1 \sin\delta & +2\mathfrak{X}_1\mathfrak{X}_2 \cos\delta \end{pmatrix} \begin{pmatrix} d\mathfrak{X}_1 \\ d\mathfrak{X}_2 \\ d\delta \end{pmatrix} = \delta u_1 \cdot \begin{pmatrix} 0 \\ +2\mathfrak{X}_1\mathfrak{X}_2 \sin\delta \\ -2\mathfrak{X}_1\mathfrak{X}_2 \cos\delta \end{pmatrix}$$

etc., which by matrix inversion give

[77] Notice in connection with (231) that

$$[L_1, [L_2, \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}]] = \mathbb{L}_2 \mathbb{L}_1 \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \quad : \quad \text{order reversed on the right}$$

and that $\mathbb{L}_2 \mathbb{L}_1 - \mathbb{L}_1 \mathbb{L}_2 = \mathbb{L}_3$, etc. involve "reversed commutators."

$$\begin{pmatrix} d\mathcal{X}_1 \\ d\mathcal{X}_2 \\ d\delta \end{pmatrix} = \begin{cases} \delta u_1 \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ \\ \tfrac{1}{2}\delta u_2 \cdot \begin{pmatrix} -\mathcal{X}_2 \sin\delta \\ +\mathcal{X}_1 \sin\delta \\ (\mathcal{X}_1\mathcal{X}_2)^{-1}(\mathcal{X}_1^2 - \mathcal{X}_2^2)\cos\delta \end{pmatrix} \\ \\ \tfrac{1}{2}\delta u_2 \cdot \begin{pmatrix} +\mathcal{X}_2 \cos\delta \\ -\mathcal{X}_1 \cos\delta \\ (\mathcal{X}_1\mathcal{X}_2)^{-1}(\mathcal{X}_1^2 - \mathcal{X}_2^2)\sin\delta \end{pmatrix} \end{cases} \tag{233}$$

Equations (229) expose the detailed mechanism by which the observables $L_\mu(x_1, p_1, x_2, p_2)$ manage to generate within 4-dimensional phase space a representation of $O(3)$ —a representation which becomes covert when projected onto 2-dimensional configuration space, where it assumes the form

$$\text{orbital ellipse} \longrightarrow \text{orbital ellipse}$$

but which (see again Figure 3) the "mechanical Stokes parameters" render explicit.

It is, from one point of view, remarkable that L_1, L_2 and L_3 are such happy bedfellows as they have shown themselves to be, for while

$$L_3(x_1, p_1, x_2, p_2) = \tfrac{1}{2}(\text{angular momentum})$$

is, by Noether's theorem, reflective of the rotational symmetry of the system, the non-linear momentum-dependence of L_1 and L_2 is of such a nature as to render those conserved observables fundamentally "non-Noetherean."[78] Notice also that while

- the flying $\mathbf{E}(\mathrm{t})$-vector presented by a monochromatic beam
- the position vector $\boldsymbol{x}(t)$ of an oscillatory mass point

move identically

$$\mathbf{E}(t) = \begin{pmatrix} \mathcal{E}_1 \cos(\omega t + \delta_1) \\ \mathcal{E}_2 \cos(\omega t + \delta_2) \end{pmatrix}$$

$$\updownarrow \boxed{\text{compare}}$$

$$\boldsymbol{x}(t) = \begin{pmatrix} \mathcal{X}_1 \cos(\omega t + \delta_1) \\ \mathcal{X}_2 \cos(\omega t + \delta_2) \end{pmatrix}$$

and therefore support the same Stokes formalism, only the latter moves *in response to the laws of mechanics*; only the latter permits Stokes' formalism to be associated with the Hamiltonian apparatus of canonical transform theory, which possesses no optical counterpart.[79]

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[78] See the discussion on p. 18 in "Kepler problem by descent from the Euler problem" (1996).

[79] "Hamiltonian optics" springs, of course, to mind, but does not fill the bill.

CANONICAL TRANSFORM THEORY IN THE KEPLEREAN CASE

As has been already remarked, and as *Mathematica* would quickly confirm, the Keplerean observables J_μ introduced at (218) mimic the algebraic and bracket properties (219) and (220) of their oscillatory counterparts L_μ

$$J_0^2 - J_1^2 - J_2^2 - J_3^2 = 0 \quad \text{and} \quad \begin{aligned}[J_1, J_2] &= J_3\\ [J_2, J_3] &= J_1\\ [J_3, J_1] &= J_2\end{aligned} \tag{234}$$

$$\Downarrow$$

$$[J_0, J_1] = [J_0, J_2] = [J_0, J_3] = 0$$

but their Cartesian descriptions are relatively complicated, and so also are the transformations which they generate; the Keplerean counterparts

$$\begin{pmatrix} \delta x_1 \\ \delta p_1 \\ \delta x_2 \\ \delta p_2 \end{pmatrix} = \delta u_\mu \cdot [J_\mu, \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix}] \tag{235}$$

of (229) are in fact—except in the case J_3—so complicated that it would serve no useful purpose to write them out; I must be content merely to indicate *how* they might be obtained (how, that is to say, the results burbed out by *Mathematica* might most usefully be organized and understood). We proceed from the observation that the definitions (218) can be notated

$$\begin{aligned}
J_0 &= k\sqrt{\tfrac{1}{2}m}(-H)^{-\frac{1}{2}}\\
J_1 &= \sqrt{\tfrac{1}{2}m}(-H)^{-\frac{1}{2}}B_1\\
J_2 &= \sqrt{\tfrac{1}{2}m}(-H)^{-\frac{1}{2}}B_2\\
J_3 &= x_1p_2 - x_2p_1
\end{aligned}$$

and that $[(-H)^{-\frac{1}{2}}B, z] = \frac{1}{2}(-H)^{-\frac{3}{2}}B \cdot [H, z] + (-H)^{-\frac{1}{2}} \cdot [B, z]$, to which we bring the following computed information:

$$\begin{aligned}
[H, x_1] &= -\tfrac{1}{m}p_1\\
[H, p_1] &= kx_1/r^3 \quad : \quad r \equiv \sqrt{x_1^2 + x_2^2}\\
[H, x_2] &= -\tfrac{1}{m}p_2\\
[H, p_2] &= kx_2/r^3
\end{aligned}$$

$$\begin{aligned}
[B_1, x_1] &= \tfrac{1}{m}x_2p_2 & [B_2, x_1] &= \tfrac{1}{m}(x_1p_2 - 2x_2p_1)\\
[B_1, p_1] &= \tfrac{1}{m}p_2^2 - kx_2^2/r^3 & [B_2, p_1] &= -\tfrac{1}{m}p_1p_2 + kx_1x_2/r^3\\
[B_1, x_2] &= \tfrac{1}{m}(x_2p_1 - 2x_1p_2) & [B_2, x_2] &= \tfrac{1}{m}x_1p_1\\
[B_1, p_2] &= -\tfrac{1}{m}p_1p_2 + kx_1x_2/r^3 & [B_2, p_2] &= \tfrac{1}{m}p_1^2 - kx_1^2/r^3
\end{aligned}$$

This data could be used to construct equations of the type (235) in the cases $\mu = 0, 1$ and 2, but in the case $\mu = 3$ the results are in simple and immediate:

$$\begin{pmatrix} \delta x_1 \\ \delta p_1 \\ \delta x_2 \\ \delta p_2 \end{pmatrix} = \delta u_3 \cdot [J_3, \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix}] = \begin{pmatrix} +x_2 \\ +p_2 \\ -x_1 \\ -p_1 \end{pmatrix}$$

Complications beset the Keplerean theory also for this second reason: the equations (222) which describe the "mechanical Stokes observables" are more intricate than their oscillatory counterparts (221). But from (235) it follows readily that

$$\left.\begin{aligned}
&[J_0, S_\mu] = 0 \quad : \quad \mu = 0, 1, 2, 3 \\
\\
&[J_1, S_0] = -\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_2J_3 \\
&[J_1, S_1] = -\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_2J_3 \\
&[J_1, S_2] = +\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_1J_3 \\
&[J_1, S_3] = -\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_0J_2 \\
\\
&[J_2, S_0] = +\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_1J_3 \\
&[J_2, S_1] = -\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_1J_3 \\
&[J_2, S_2] = -\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_2J_3 \\
&[J_2, S_3] = +\tfrac{1}{m^2k^2} J_0^2 \cdot 2J_0J_1 \\
\\
&[J_3, S_0] = 0 \\
&[J_3, S_1] = +\tfrac{1}{m^2k^2} J_0^2 \cdot 4J_1J_2 \\
&[J_3, S_2] = -\tfrac{1}{m^2k^2} J_0^2 \cdot 2(J_1^2 - J_2^2) \\
&[J_3, S_3] = 0
\end{aligned}\right\} \tag{236}$$

The pattern—such as it is—is disappointingly uninformative, but one fact at least is clear: the Stokes observables $\{S_0, S_1, S_2, S_3\}$ respond more simply to J_μ-generated canonical transformations

$$\begin{pmatrix} \delta S_0 \\ \delta S_1 \\ \delta S_2 \\ \delta S_3 \end{pmatrix} = \delta u_\mu \cdot [J_\mu, \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}]$$

than do the phase coordinates $\{x_1, p_1, x_2, p_2\}$ themselves. From (236) follow finally the Stokes-Stokes brackets

$$\left.\begin{aligned}
[S_0,S_1] &= \left(\tfrac{1}{m^2k^2}J_0^2\right)^2\cdot 8J_3J_1J_2 &&= +2S_2S_3/J_0\\
[S_0,S_2] &= \left(\tfrac{1}{m^2k^2}J_0^2\right)^2\cdot 4J_3(J_2^2-J_1^2) &&= -2S_3S_1/J_0\\
[S_0,S_3] &= 0\\
\\
[S_1,S_2] &= \left(\tfrac{1}{m^2k^2}J_0^2\right)^2\cdot 4J_3(J_1^2+J_2^2)\\
[S_2,S_3] &= \left(\tfrac{1}{m^2k^2}J_0^2\right)^2\cdot 4J_0(J_1^2-J_2^2)\\
[S_3,S_1] &= \left(\tfrac{1}{m^2k^2}J_0^2\right)^2\cdot 8J_0J_1J_2
\end{aligned}\right\} \tag{237}$$

I would ask no person actually to *compute* brackets of the types $[J_\mu, S_\nu]$ and $[S_\mu, S_\nu]$, but *Mathematica* finds the assignment no cause for complaint, and has independently varified each of the statements (236/7).

Equations (237) are notable for the complexity of the expressions which appear on the right. Had those expressions been of the form $\sum c_{\mu\nu}{}^\alpha S_\alpha$ then (237) would have opened a Lie-theoretic door, but the facts are otherwise. We conclude that in Kepler theory the Stokes observables (222) serve usefully to describe the orbital figure, but do not participate directly in the group theory of the problem, which remains the special province of observables J_μ which are themselves but thinly disguised variants of $\mathbf{K}$ and L_3. This is in stark contrast to the oscillatory situation, where $S_\mu \sim J_\mu$ do "participate directly in the group theory." But in the latter context a different kind of crookedness intrudes: one still has $S_3 \sim L_3$, but in the absence of an oscillatory Runge-Lenz vector it becomes impossible to write $S_{1,2} \sim \mathbf{K}$. Though J_μ^{Kepler} and $J_\mu^{\text{oscillator}}$ share important properties, their dynamical roots are seemingly quite distinct.

18. How Gibbs might have approached the isotropic oscillator problem. Gibb's magically efficient treatment of the Kepler problem (summarized near the end of §14) was intended mainly to demonstrate the utility of his "vector analysis." The question arises: Can Gibbs' method be brought to bear also on other central force problems? A quick check shows that the argument which yielded the Runge-Lenz vector $\mathbf{K}$ as a natural vector-valued constant of integration works *only* in the Keplerean case $\boldsymbol{F} = -k\frac{1}{r^3}\boldsymbol{x}$. So the question becomes: Can a variant of Gibbs' method be devised which works with other central forces? Which works, in particular, in the oscillatory case $\boldsymbol{F} = -k\boldsymbol{x}$, which we (like Bertrand before us) have seen to be in several respects the Kepler problem's most natural companion?

Gibbs' argument makes essential use of the "cross product," and in that respect gains leverage from the circumstance that the orbital plane resides in 3-space. Let us agree, therefore, to work 3-dimensionally. How to proceed? I take my cue from a classic paper by D. M. Fradkin,[80] cited by Goldstein as a

[80] "Existence of the dynamic symmmetries O_4 and SU_3 for all classical central potential problems," Progress of Theoretical Physics **37**, 798 (1967).

source for discussion of "generalized Runge-Lenz vectors," though the author's intention actually lay elsewhere, and was rather more grand.

Gibbs' *Vector Analysis* provides no index, but I am satisfied that such an index, if it existed, would contain no entry at the word "matrix." Neither the word, nor the notation, nor more than a hint of the idea are to be found within those famous pages.[81] In its stead one encounters the "dyadic," to the theory and applications of which he devotes the last third of his text. Given a pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, Gibbs writes $\boldsymbol{a}\,\boldsymbol{b}$ where Dirac writes $|a)(b|$ and we might write $\boldsymbol{a}\otimes\boldsymbol{b}$ to describe the tensor (or "outer") product, which Gibbs calls a "dyad." He adopts the terminology

$$\text{dyadic} = \text{dyad} + \text{dyad} + \cdots$$

and other quaint locutions with which we need not concern ourselves. Dyadics are by nature operators, and acquire matrix-valued representations when referred to a basis

$$|a)(b| = \sum\sum a_i b_j |i)(j| \quad \text{with} \quad a_i \equiv (i\,|a),\, b_j \equiv (b|j)$$

Gibb's scheme was in many ways anticipatory of Dirac's (but much less elegantly rendered), and one does still encounter occasional modern proponents.[82] But I have progressed already far enough into the subject to say what I have to say, and feel no temptation to proceed farther.

Given the system $m\ddot{\boldsymbol{x}} = -k\boldsymbol{x}$, Gibbs might (I suggest, though in the concluding sections of his book—where he treats precisely this system—he evidently didn't) find it natural to introduce the dyadic

$$\mathcal{K} \equiv \tfrac{1}{2m}\boldsymbol{p}\otimes\boldsymbol{p} + \tfrac{1}{2}k\boldsymbol{x}\otimes\boldsymbol{x} \tag{238.1}$$

and to observe that

$$\begin{aligned}\tfrac{d}{dt}\mathcal{K} &= \quad \tfrac{1}{2m}\,\dot{\boldsymbol{p}}\otimes\boldsymbol{p} + \tfrac{1}{2m}\boldsymbol{p}\otimes\dot{\boldsymbol{p}} + \tfrac{1}{2}k\,\dot{\boldsymbol{x}}\otimes\boldsymbol{x} + \tfrac{1}{2}k\,\boldsymbol{x}\otimes\dot{\boldsymbol{x}}\\ &= -\tfrac{k}{2m}\boldsymbol{x}\otimes\boldsymbol{p} - \tfrac{k}{2m}\boldsymbol{p}\otimes\boldsymbol{x} + \tfrac{k}{2m}\,\boldsymbol{p}\otimes\boldsymbol{x} + \tfrac{k}{2m}\,\boldsymbol{x}\otimes\boldsymbol{p} = 0\end{aligned}$$

---

[81] For reasons which are, I think, intelligible. Gibbs, the Yale Yankee, was at pains to separate himself from the "Quaternionic Wars" which had raged for more than sixty years on the other side of the Atlantic, and which had recently established a secure beachhead at Harvard (where his co-author had done his undergraduate work). Matrix theory derives from a memoir written by Arthur Cayley in 1858, and was cultivated at Harvard by Benjamin Pierce and his son Charles Sanders Pierce. I suspect that Gibbs found it all too easy to associate matrix theory (which Born and Heisenberg as late as 1925 considered esoteric) with the body of mathematics he sought to supplant.

[82] See, for example, Goldstein's §5-2.

according to which $\mathcal{K}$ is a dyadic-valued (non-Noetherean) constant of the motion. In matrix notation we have $\dot{\mathbb{K}} = \mathbb{O}$ with

$$\mathbb{K} \equiv \tfrac{1}{2m}\begin{pmatrix} p_1p_1 & p_1p_2 & p_1p_3 \\ p_2p_1 & p_2p_2 & p_2p_3 \\ p_3p_1 & p_3p_2 & p_3p_3 \end{pmatrix} + \tfrac{1}{2}k\begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{pmatrix} \tag{238.2}$$

Whether one works from (238.1) or from (238.2), it follows as an immediate consequence of $\boldsymbol{x}\cdot(\boldsymbol{x}\times\boldsymbol{p}) = \boldsymbol{p}\cdot(\boldsymbol{x}\times\boldsymbol{p}) = \mathbf{0}$ that

$$\mathbb{K}\mathbf{L} = \mathbf{0} \tag{239}$$

i.e., that the angular momentum vector $\mathbf{L}$ is an *eigenvector* of $\mathbb{K}$, with null eigenvalue. From the manifest real symmetry of $\mathbb{K}$ we know it to be the case that

- All eigenvalues of $\mathbb{K}$ are necessarily real, and
- Eigenvectors associated with distinct eigenvalues are necessarily orthogonal.

Let the $\mathbb{K}$-spectrum be notated $\{0, \lambda_1, \lambda_2\}$. *Mathematica* provides complicated expressions for $\lambda_{1,2}$ from which follow

$$\begin{aligned}
\text{tr}\,\mathbb{K} = \lambda_1 + \lambda_2 &= \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \tfrac{1}{2}k\boldsymbol{x}\cdot\boldsymbol{x} \\
&= H \;:\; \text{the oscillator Hamiltonian} \\
\det\mathbb{K} = \lambda_1\cdot\lambda_2\cdot 0 &= 0 \\
\lambda_1\cdot\lambda_2 \quad &= \tfrac{k}{4m}\big\{(\boldsymbol{p}\cdot\boldsymbol{p})(\boldsymbol{x}\cdot\boldsymbol{x}) - (\boldsymbol{p}\cdot\boldsymbol{x})^2\big\} \\
&= \tfrac{k}{4m}(\boldsymbol{x}\times\boldsymbol{p})\cdot(\boldsymbol{x}\times\boldsymbol{p}) \\
&= k\ell^2/4m
\end{aligned}$$

To simplify progress beyond this point (i.e., to obtain expressions simple enough to comprehend) let us now suppose the orbit to lie in the $\{x_1, x_2\}$-plane; then $x_3 = p_3 = 0$, and we have

$$\mathbb{K} = \tfrac{1}{2m}\begin{pmatrix} p_1p_1 & p_1p_2 & 0 \\ p_2p_1 & p_2p_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tfrac{1}{2}k\begin{pmatrix} x_1x_1 & x_1x_2 & 0 \\ x_2x_1 & x_2x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} H_1 & H_0 & 0 \\ H_0 & H_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 0 \\ 0 \\ x_1p_2 - x_2p_1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \ell \end{pmatrix}$$

Borrowing our methods now from §1, we observe that the eigenvalues of

$$\mathbb{H} \equiv \begin{pmatrix} H_1 & H_0 \\ H_0 & H_2 \end{pmatrix} \tag{240}$$

can be described

$$\begin{aligned}
\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} &= \tfrac{1}{2}E \pm \sqrt{(\tfrac{1}{2}D)^2 + H_0^2} \qquad \text{with} \qquad \left\{\begin{matrix} E \equiv H_1 + H_2 \\ D \equiv H_1 - H_2 \end{matrix}\right. \\
&= \tfrac{1}{2}E \pm \sqrt{(\tfrac{1}{2}E)^2 - (H_1H_2 - H_0^2)} \\
&= \tfrac{1}{2}E \pm \sqrt{(\tfrac{1}{2}E)^2 - k\ell^2/4m}
\end{aligned} \tag{241}$$

in terms of which we have[83]

$$\left.\begin{aligned} H_1 &= \tfrac{\lambda_1+\lambda_2}{2} + \tfrac{\lambda_1-\lambda_2}{2}\cos 2\psi \\ H_2 &= \tfrac{\lambda_1+\lambda_2}{2} - \tfrac{\lambda_1-\lambda_2}{2}\cos 2\psi \\ H_0 &= \qquad\quad\;\; \tfrac{\lambda_1-\lambda_2}{2}\sin 2\psi \end{aligned}\right\} \tag{242}$$

and the normalized eigenvectors become very easy to describe:

$$\mathbb{H}\begin{pmatrix} \cos\psi \\ +\sin\psi \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\psi \\ +\sin\psi \end{pmatrix} \quad \text{and} \quad \mathbb{H}\begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix} = \lambda_2 \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}$$

In the latter connection we notice that

$$\begin{aligned} \cos\psi &= \sqrt{\tfrac{1}{2}(1+\cos 2\psi)} \\ \sin\psi &= \sqrt{\tfrac{1}{2}(1-\cos 2\psi)} \end{aligned}$$

$$\cos 2\psi = \frac{D}{\lambda_1 - \lambda_2} = \frac{D}{\sqrt{D^2 + 4H_0^2}} = \frac{D}{\sqrt{E^2 - k\ell^2/m}}$$

Note also the striking simplifications which come about when $\mathbb{H}$ is delivered to us in "already diagonalized" form: $H_0 = 0$.

Look now to the equations which describe dynamical flow in the phase space of a 2-dimensional isotropic oscillator

$$\left.\begin{aligned} x_1(t) &= \qquad\;\; \mathcal{X}_1\cos(\omega t + \delta_1) \\ p_1(t) &= -m\omega\mathcal{X}_1\sin(\omega t + \delta_1) \\ x_2(t) &= \qquad\;\; \mathcal{X}_2\cos(\omega t + \delta_2) \\ p_2(t) &= -m\omega\mathcal{X}_2\sin(\omega t + \delta_2) \end{aligned}\right\} \tag{243}$$

By quick computation

$$\begin{aligned} H_1 &= \tfrac{1}{2}k\cdot\mathcal{X}_1^2 \\ H_2 &= \tfrac{1}{2}k\cdot\mathcal{X}_2^2 \\ H_0 &= \tfrac{1}{2}k\cdot\mathcal{X}_1\mathcal{X}_2\cos\delta \quad \text{where again:} \quad \delta \equiv \delta_2 - \delta_1 \\ \sqrt{H_1H_2 - H_0^2} &= \tfrac{1}{2}k\cdot\mathcal{X}_1\mathcal{X}_2\sin\delta \end{aligned}$$

and we have been led to the brink of a "reinvention" of the mechanical Stokes parameters (159):

$$\left.\begin{aligned} S_0 &= \mathcal{X}_1^2 + \mathcal{X}_2^2 &&= \tfrac{2}{m\omega^2}\cdot(H_1 + H_2) \\ S_1 &= \mathcal{X}_1^2 - \mathcal{X}_2^2 &&= \tfrac{2}{m\omega^2}\cdot(H_1 - H_2) \\ S_2 &= 2\mathcal{X}_1\mathcal{X}_2\cos\delta &&= \tfrac{2}{m\omega^2}\cdot 2H_0 \\ S_3 &= 2\mathcal{X}_1\mathcal{X}_2\sin\delta &&= \tfrac{2}{m\omega^2}\cdot 2\sqrt{H_1H_2 - H_0^2} \end{aligned}\right\} \tag{244}$$

[83] See again Figure 1.

But we learned already in §1 that if $\sqrt{uv-w^2}$ is real then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u & w \\ w & v \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1$$

describes a centered ellipse of size

$$S_0 = \mathcal{X}_1^2 + \mathcal{X}_2^2 = \tfrac{1}{uv-w^2} \cdot (u+v)$$

of which

$$\begin{aligned} \tfrac{1}{uv-w^2} \cdot (u-v) &= S_0 \cos 2\chi \cos 2\psi \equiv S_1 \\ \tfrac{1}{uv-w^2} \cdot \quad 2w \quad &= S_0 \cos 2\chi \sin 2\psi \equiv S_2 \\ \tfrac{1}{uv-w^2} \cdot 2\sqrt{uv-w^2} &= S_0 \sin 2\chi \qquad\quad \equiv S_3 \end{aligned}$$

serve, after the manner indicated in Figure 3, to describe the orientation and figure. Comparison with (244) leads us to set

$$\begin{aligned} \tfrac{2}{m\omega^2} H_1 &= \tfrac{u}{uv-w^2} \\ \tfrac{2}{m\omega^2} H_2 &= \tfrac{v}{uv-w^2} \\ \tfrac{2}{m\omega^2} H_0 &= \tfrac{w}{uv-w^2} \end{aligned}$$

or—which is by $uv - w^2 = \left[\left(\tfrac{2}{m\omega^2}\right)^2 (H_1 H_2 - H_0^2)\right]^{-1}$ the same—to write

$$\begin{pmatrix} u & w \\ w & v \end{pmatrix} = \tfrac{m\omega^2}{2} \tfrac{1}{\det \mathbb{H}} \begin{pmatrix} H_1 & H_0 \\ H_0 & H_2 \end{pmatrix}$$

and brings us to the conclusion that the elliptical orbit of the oscillator can be described

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} H_1 & H_0 \\ H_0 & H_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tfrac{2}{m\omega^2} \det \mathbb{H}$$

In the 3-dimensional case

$$\boldsymbol{x}^{\mathsf{T}} \mathbb{K}\, \boldsymbol{x} = \tfrac{2}{m\omega^2} (\text{product of non-zero eigenvalues of } \mathbb{K}) \tag{245}$$

serves to describe the orbit as a *plane curve in 3-space*.[84]

To summarize: the methods which led Gibbs[85] so effortlessly/naturally to the "invention" of the Runge-Lenz vector **K** and to the Keplerean orbit (191) fail when applied to the isotropic oscillator. But a closely related method, which makes use of material developed in the final (dyadic) chapters of *Vector Analysis*, can be devised. It emerges that the oscillatory counterpart of the conserved vector **K** is a conserved dyadic $\mathcal{K}$, representable as a symmetric

[84] The truth of this proposition was known already to Fradkin, but his argument is unconvincing; he provides only the illustrative details of a very simple special case.

[85] See again pp. 77–78.

matrix $\mathbb{K}$ of the specialized design (238.2). One looks (most efficiently with the aid of Mohr's construction: (242)) to the spectral properties of $\mathcal{K}$, as summarized in the "spectral resolution" which Gibbs (in clear anticipation of Dirac) might notate

$$\mathcal{K} = \mathbf{K}_1\lambda_1\mathbf{K}_1 + \mathbf{K}_2\lambda_2\mathbf{K}_2 + \mathbf{L}\,0\,\mathbf{L}$$

The orthonormal eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$ identify the principal axes of the orbital ellipse, and the eigenvalues assemble themselves (almost spontaneously) into Stokes parameters descriptive of the orbital figure.

The novelty of the preceding discussion resides entirely in the arrangement of the points of emphasis; we have simply poured old material (see again, for example, (163/165)) into a Gibbs-shaped bottle. It would be interesting on another occasion to devise a formalism which provides a *unified* account of the Kepler/oscillator problems, and to make make more explicit the details of their "parabolic interconvertability."

PART III: APPLICATIONS TO QUANTUM MECHANICS

I have on two recent occasions—at length in "Reduced Kepler problem in elliptic coordinates" (1999), and more succinctly in "Classical/quantum theory of 2-dimensional hydrogen" (1999)—discussed comparatively the quantum mechanics (spectrum and eigenfunctions) of the Kepler and isotropic oscillator problems, in terms which emphasize the "parabolic equivalence" of those two systems, and which are also in other ways consonant with the present discussion. I do not propose to repeat that material here. I inquire instead into the placement of the "orbital" concept within quantum mechanics—a concept which Bohr injected (borrowed from classical mechanics) and which Schrödinger (or so it is commonly imagined) effectively displaced—and into the emergence of Stokes parameters as natural descriptors of the "quantum orbits" presented by the two systems in which (quantum echo of Bertrand) we have special interest.

19. "Orbits of the moments" for the isotropic quantum oscillator. We look to the system

$$\mathsf{H} = \tfrac{1}{2m}(\mathsf{p}_1^2 + \mathsf{p}_2^2) + \tfrac{1}{2}k(\mathsf{x}_1^2 + \mathsf{x}_2^2) \tag{246}$$

If A refers to a time-independent observable, and $|\psi)$ to the state of the system, then

$$\langle \mathsf{A}\rangle \equiv (\psi|\mathsf{A}|\psi) = \text{expected average of many } \mathsf{A}\text{-measurements}$$

and—independently of whether one has elected to work in the Schrödinger picture or the Heisenberg picture—one has

$$\tfrac{d}{dt}\langle \mathsf{A}\rangle = \tfrac{1}{i\hbar}\langle \mathsf{AH} - \mathsf{HA}\rangle \tag{247}$$

The commutator on the right is especially easy to evaluate when $\mathbf{H}$ is of the quadratic design (246); as particular instances of (247) one has

$$\left.\begin{aligned} \tfrac{d}{dt}\langle\mathsf{x}_1\rangle &= \ \tfrac{1}{m}\langle\mathsf{p}_1\rangle \\ \tfrac{d}{dt}\langle\mathsf{p}_1\rangle &= -k\langle\mathsf{x}_1\rangle \\ \tfrac{d}{dt}\langle\mathsf{x}_2\rangle &= \ \tfrac{1}{m}\langle\mathsf{p}_2\rangle \\ \tfrac{d}{dt}\langle\mathsf{p}_2\rangle &= -k\langle\mathsf{x}_2\rangle \end{aligned}\right\} \tag{248}$$

Ehrenfest's theorem asserts[86] that the motion of the moments on the left is *approximately* classical in *all* cases, but in the oscillatory case (246) it is *exactly* classical *for all states* $|\psi)$. This development is as remarkable as it is elementary, and merits several kinds of commentary.

If $|\psi)$ is an energy eigenstate[87]

$$|\psi) = |E)e^{-\frac{i}{\hbar}Et}$$

then (for separate reasons on right and left)

$$\tfrac{d}{dt}\langle\mathbf{A}\rangle = \tfrac{1}{i\hbar}\langle\mathbf{AH}-\mathbf{HA}\rangle \quad \text{becomes} \quad 0=0$$

which is, however, not so uninteresting as it might appear: the remark vividly underscores the force of the term "time-independent quantum mechanics," and demonstrates that to obtain motion rather than stasis in the observable output of that theory the wave function must present a non-trivial *superposition* of energy eigenstates:

$$|\psi) = c_1|E_1)e^{-\frac{i}{\hbar}E_1t} + c_2|E_2)e^{-\frac{i}{\hbar}E_2t} \quad : \quad E_1 \neq E_2 \tag{249}$$

Observable motion, in other words, is an attribute not of energy eigenstates but of wave *packets*. And in the instance (248) we have

$$(E|\mathsf{x}_1|E) = (E|\mathsf{p}_1|E) = (E|\mathsf{x}_2|E) = (E|\mathsf{p}_2|E) = 0 \tag{250}$$

for all oscillator eigenstates.

The equations (248) are readily *decoupled by differentiation*; one finds that each of the first moments $\langle\mathsf{x}_1\rangle$, $\langle\mathsf{p}_1\rangle$, $\langle\mathsf{x}_2\rangle$ and $\langle\mathsf{p}_2\rangle$ moves in such a way as to satisfy

$$\tfrac{d^2}{dt^2}\langle\bullet\rangle = -\omega^2\langle\bullet\rangle \tag{251}$$

with $\omega^2 \equiv k/m$. Whether we work from (248) or from (251), we are—precisely as in classical dynamics—led to solutions of the form

$$\left.\begin{aligned} \langle\mathsf{x}_1\rangle_t &= \qquad\ \langle\mathsf{x}_1\rangle_0\cos\omega t + \tfrac{1}{m\omega}\langle\mathsf{p}_1\rangle_0\sin\omega t \\ \langle\mathsf{p}_1\rangle_t &= -m\omega\langle\mathsf{x}_1\rangle_0\sin\omega t + \qquad \langle\mathsf{p}_1\rangle_0\cos\omega t \\ \langle\mathsf{x}_2\rangle_t &= \qquad\ \langle\mathsf{x}_2\rangle_0\cos\omega t + \tfrac{1}{m\omega}\langle\mathsf{p}_2\rangle_0\sin\omega t \\ \langle\mathsf{p}_2\rangle_t &= -m\omega\langle\mathsf{x}_2\rangle_0\sin\omega t + \qquad \langle\mathsf{p}_2\rangle_0\cos\omega t \end{aligned}\right\} \tag{252}$$

[86] See "Status and some ramifications of Ehrenfest's theorem" (1998).

[87] I slip here into the Schrödinger picture, where I will find it convenient to remain.

But no longer are we free (as in classical dynamics we were) to assign values independently to $\langle \mathsf{x}_1\rangle_0$, $\langle \mathsf{p}_1\rangle_0$, $\langle \mathsf{x}_2\rangle_0$ and $\langle \mathsf{p}_2\rangle_0$; those—correlated!—numbers are implicit in the specification of $|\psi)_0$. The moving quartet

$$\big\{\langle \mathsf{x}_1\rangle_t, \langle \mathsf{p}_1\rangle_t, \langle \mathsf{x}_2\rangle_t, \langle \mathsf{p}_2\rangle_t\big\}$$

traces what we may agree to call a “quantum flow line” in phase space,[88] while

$$\big\{\langle \mathsf{x}_1\rangle_t, \langle \mathsf{x}_2\rangle_t\big\}$$

traces a “quantum orbit” in configuration space. It follows from what has been said that the latter curves are centered ellipses—curves describable by a set of “quantum Stokes parameters” S_μ. The figure of such an ellipse is implicit in the specification of $|\psi)_0$. We will not rest until we possess formulæ which

$$\text{describe } S_\mu \text{ directly in terms of } |\psi)_0$$

But look now to the motion of the second moments $\langle \mathsf{x}_1^2\rangle$, $\langle \mathsf{p}_1^2\rangle$, $\langle \mathsf{x}_2^2\rangle$ and $\langle \mathsf{p}_2^2\rangle$, which in consequence of (250) are also *centered* second moments:

$$\langle(\mathsf{x}_1 - \langle \mathsf{x}_1\rangle)^2\rangle = \langle \mathsf{x}_1^2\rangle, \text{ etc.}$$

Drawing upon $[\mathsf{x},\mathsf{p}] = i\hbar$ and the fundamental identity $[\mathsf{AB},\mathsf{C}] = \mathsf{A}[\mathsf{B},\mathsf{C}]+[\mathsf{A},\mathsf{B}]\mathsf{C}$ we find

$$[\mathsf{x}^2,\mathsf{p}^2] = 2i\hbar(\mathsf{x}\,\mathsf{p} + \mathsf{p}\,\mathsf{x}) \tag{253.1}$$

to which (in order to achieve algebraic closure) we must adjoin

$$[\mathsf{x}^2,\mathsf{x}\,\mathsf{p} + \mathsf{p}\,\mathsf{x}] = +4i\hbar\,\mathsf{x}^2 \tag{253.2}$$
$$[\mathsf{p}^2,\mathsf{x}\,\mathsf{p} + \mathsf{p}\,\mathsf{x}] = -4i\hbar\,\mathsf{p}^2 \tag{253.3}$$

Returning with this information to (247) we obtain

$$\left.\begin{aligned}\tfrac{d}{dt}\langle \mathsf{x}^2\rangle &= +\tfrac{1}{2m}\cdot 2\langle \mathsf{xp} + \mathsf{xp}\rangle\\ \tfrac{d}{dt}\langle \mathsf{p}^2\rangle &= -\tfrac{1}{2}k\cdot 2\langle \mathsf{xp} + \mathsf{xp}\rangle\end{aligned}\right\} \tag{253.4}$$

It follows (by way of a consistency check) that

$$\tfrac{d}{dt}\langle \tfrac{1}{2m}\mathsf{p}^2 + \tfrac{1}{2}k\mathsf{x}^2\rangle = 0 \tag{253.5}$$

[88] Note, however, that, in consequence of (250), if $|\psi)$ is an energy eigenstate then we are back again to stasis: the “flow line” has become degenerate—a “stagnation point” at the origin. This remark pertains, I would emphasize, not just to the ground state, but to *any* energy eigenstate, however great may be the value of E.

and more interestingly that

$$\left.\begin{aligned}\tfrac{d}{dt}\langle\tfrac{1}{2m}\mathsf{p}^2-\tfrac{1}{2}k\mathsf{x}^2\rangle &= -\omega^2\langle\mathsf{xp}+\mathsf{xp}\rangle\\ \tfrac{d}{dt}\langle\mathsf{xp}+\mathsf{xp}\rangle &= 4\langle\tfrac{1}{2m}\mathsf{p}^2-\tfrac{1}{2}k\mathsf{x}^2\rangle\end{aligned}\right\}\tag{253.6}$$

It follows from this last pair of equations that both $\frac{1}{2m}\mathsf{p}^2-\frac{1}{2}k\mathsf{x}^2$ and $\mathsf{xp}+\mathsf{xp}$ satisfy

$$\tfrac{d^2}{dt^2}\langle\bullet\rangle = -(2\omega)^2\langle\bullet\rangle\tag{253.7}$$

which is to say: they oscillate with doubled frequency. Writing

$$\begin{aligned}\mathsf{x}^2 &= 2\tfrac{1}{k}\cdot\tfrac{1}{2}\big[\big(\tfrac{1}{2m}\mathsf{p}^2+\tfrac{1}{2}k\mathsf{x}^2\big)-\big(\tfrac{1}{2m}\mathsf{p}^2-\tfrac{1}{2}k\mathsf{x}^2\big)\big]\\ &= \tfrac{1}{2}\big(\mathsf{x}^2+\tfrac{1}{km}\mathsf{p}^2\big)+\tfrac{1}{2}\big(\mathsf{x}^2-\tfrac{1}{km}\mathsf{p}^2\big)\end{aligned}$$

$$\begin{aligned}\mathsf{p}^2 &= 2m\cdot\tfrac{1}{2}\big[\big(\tfrac{1}{2m}\mathsf{p}^2+\tfrac{1}{2}k\mathsf{x}^2\big)+\big(\tfrac{1}{2m}\mathsf{p}^2-\tfrac{1}{2}k\mathsf{x}^2\big)\big]\\ &= \tfrac{1}{2}\big(\mathsf{p}^2+km\mathsf{x}^2\big)+\tfrac{1}{2}\big(\mathsf{p}^2-km\mathsf{x}^2\big)\end{aligned}$$

we are led to the conclusion that

$$\left.\begin{aligned}\langle\mathsf{x}^2\rangle_t &= \big\langle\tfrac{1}{2}\big(\mathsf{x}^2+\tfrac{1}{km}\mathsf{p}^2\big)\big\rangle_0+\big\langle\tfrac{1}{2}\big(\mathsf{x}^2-\tfrac{1}{km}\mathsf{p}^2\big)\big\rangle_0\cos 2\omega t\\ &\qquad+\tfrac{1}{2m\omega}\big\langle\mathsf{xp}+\mathsf{px}\big\rangle_0\sin 2\omega t\\ \langle\mathsf{p}^2\rangle_t &= \big\langle\tfrac{1}{2}\big(\mathsf{p}^2+km\mathsf{x}^2\big)\big\rangle_0+\big\langle\tfrac{1}{2}\big(\mathsf{p}^2-km\mathsf{x}^2\big)\big\rangle_0\cos 2\omega t\\ &\qquad+\tfrac{m\omega}{2}\big\langle\mathsf{xp}+\mathsf{px}\big\rangle_0\sin 2\omega t\end{aligned}\right\}\tag{253.8}$$

By way of commentary: classically we have

$$\begin{aligned}x(t) &= x_0\cos\omega t+\tfrac{1}{m\omega}p_0\sin\omega t\\ &\Downarrow\\ x^2(t) &= \tfrac{1}{2}(x_0^2+\tfrac{1}{km}p_0^2)+\tfrac{1}{2}(x_0^2-\tfrac{1}{km}p_0^2)\cos 2\omega t+\tfrac{1}{2m\omega}(2x_0p_0)\sin 2\omega t\end{aligned}$$

and from (252) it follows that an identical statement pertains to the motion of $\langle\mathsf{x}\rangle_t^2$. While the result just achieved does resemble (253.8), and does expose the elementary origin of frequency doubling, it must be borne in mind that

$$\langle\mathsf{x}^2\rangle\neq\langle\mathsf{x}\rangle^2,\ \text{etc.}$$

so (253.8) is not a corollary of (252), but stands on its own independent legs.

It is important to appreciate that the operators x and p which enter into the statements (253) are understood in each instance to *wear the same subscripts* (all $_1$ else all $_2$).[89] Systems with two degrees of freedom supply, however, also

[89] The statements pertain with equal force therefore to the quantum theory of one-dimensional oscillators; see p. 13 of my Ehrenfest essay.[86]

a population of second moments of mixed design: $\langle \mathsf{x}_1\mathsf{x}_2\rangle$, $\langle \mathsf{x}_1\mathsf{p}_2\rangle$, $\langle \mathsf{p}_1\mathsf{x}_2\rangle$, $\langle \mathsf{p}_1\mathsf{p}_2\rangle$. Working from (246) we have

$$\left.\begin{aligned}
[\mathsf{x}_1\mathsf{x}_2, \mathsf{H}] &= +i\hbar\cdot\tfrac{1}{m}(\mathsf{x}_1\mathsf{p}_2+\mathsf{p}_1\mathsf{x}_2)\\
[\mathsf{x}_1\mathsf{p}_2, \mathsf{H}] &= i\hbar\cdot\tfrac{1}{m}(\mathsf{p}_1\mathsf{p}_2-km\mathsf{x}_1\mathsf{x}_2)\\
[\mathsf{p}_1\mathsf{x}_2, \mathsf{H}] &= i\hbar\cdot\tfrac{1}{m}(\mathsf{p}_1\mathsf{p}_2-km\mathsf{x}_1\mathsf{x}_2)\\
[\mathsf{p}_1\mathsf{p}_2, \mathsf{H}] &= -i\hbar\cdot k\ (\mathsf{x}_1\mathsf{p}_2+\mathsf{p}_1\mathsf{x}_2)
\end{aligned}\right\}\tag{254}$$

from which we conclude that

$$\left.\begin{aligned}
[(\mathsf{x}_1\mathsf{p}_2+\mathsf{p}_1\mathsf{x}_2), \mathsf{H}] &= +i\hbar\cdot 2\tfrac{1}{m}(\mathsf{p}_1\mathsf{p}_2-km\mathsf{x}_1\mathsf{x}_2)\\
[\tfrac{1}{m}(\mathsf{p}_1\mathsf{p}_2-km\mathsf{x}_1\mathsf{x}_2), \mathsf{H}] &= -i\hbar\cdot 2k\ (\mathsf{x}_1\mathsf{p}_2+\mathsf{p}_1\mathsf{x}_2)
\end{aligned}\right\}\tag{255}$$

(therefore that the two operators in question are again solutions of (253.7)) and that $\frac{1}{2m}\mathsf{p}_1\mathsf{p}_2-\frac{1}{2}k\mathsf{x}_1\mathsf{x}_2$ and $\mathsf{x}_1\mathsf{p}_2-\mathsf{x}_2\mathsf{p}_1$ are constants of oscillator motion; indeed, we have in

$$\left.\begin{aligned}
\mathsf{H}_1 &\equiv \tfrac{1}{2m}\mathsf{p}_1\mathsf{p}_1+\tfrac{1}{2}k\,\mathsf{x}_1\mathsf{x}_1\\
\mathsf{H}_2 &\equiv \tfrac{1}{2m}\mathsf{p}_2\mathsf{p}_2+\tfrac{1}{2}k\,\mathsf{x}_2\mathsf{x}_2\\
\mathsf{H}_0 &\equiv \tfrac{1}{2m}\mathsf{p}_1\mathsf{p}_2+\tfrac{1}{2}k\,\mathsf{x}_1\mathsf{x}_2\\
\mathsf{L}\ &\equiv \mathsf{x}_1\mathsf{p}_2-\mathsf{x}_2\mathsf{p}_1
\end{aligned}\right\}\tag{256}$$

quantum analogs of precisely the observables to which (in §18) we were led by "Gibbs' construction," each of which commutes with the Hamiltonian (246) of the isotropic quantum oscillator.

Proceeding now in imitation of (217) we introduce conserved observables

$$\left.\begin{aligned}
\mathsf{L}_0 &\equiv \tfrac{1}{2\omega}(\mathsf{H}_1+\mathsf{H}_2)\sim\mathsf{H}\\
\mathsf{L}_1 &\equiv \tfrac{1}{2\omega}(\mathsf{H}_1-\mathsf{H}_2)\\
\mathsf{L}_2 &\equiv \ \tfrac{1}{\omega}\ \ \mathsf{H}_0\\
\mathsf{L}_3 &\equiv \ \tfrac{1}{2}\ \ \mathsf{L}
\end{aligned}\right\}\tag{257}$$

(each of which bears the physical dimension of action) and observe that

$$\mathsf{L}_0^2-\mathsf{L}_1^2-\mathsf{L}_2^2-\mathsf{L}_3^2=(\tfrac{1}{2}\hbar)^2\,\mathsf{I}\tag{258.1}$$

$$[\mathsf{L}_1,\mathsf{L}_2]=i\hbar\mathsf{L}_3,\quad [\mathsf{L}_2,\mathsf{L}_3]=i\hbar\mathsf{L}_1,\quad [\mathsf{L}_3,\mathsf{L}_1]=i\hbar\mathsf{L}_2\tag{258.2}$$

and (therefore, redundantly with what we already know)

$$[\mathsf{L}_0,\mathsf{L}_1]=[\mathsf{L}_0,\mathsf{L}_2]=[\mathsf{L}_0,\mathsf{L}_3]=\mathbf{0}\tag{258.3}$$

These equations[90] mimic (219/220), but to establish precise contact with the related material presented in §2–8 of Jauch & Rohrlich[12] we must make some

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[90] Recall Dirac's association:

$$\text{Poisson bracket}\longleftrightarrow\tfrac{1}{i\hbar}\text{commutator}$$

slight adjustments. Let non-hermitian operators a_1, a_2, $\mathsf{b}_1 \equiv \mathsf{a}_1^+$ and $\mathsf{b}_2 \equiv \mathsf{a}_2^+$ be introduced in direct imitation (compare (151)) of (145); i.e., by constructions of the design

$$\left.\begin{aligned} \mathsf{a} &\equiv \tfrac{1}{\sqrt{2}}\big\{\sqrt{m\omega/\hbar}\cdot\mathsf{x} + i\sqrt{1/m\omega\hbar}\cdot\mathsf{p}\big\} \\ \mathsf{a}^+ \equiv \mathsf{b} &\equiv \tfrac{1}{\sqrt{2}}\big\{\sqrt{m\omega/\hbar}\cdot\mathsf{x} - i\sqrt{1/m\omega\hbar}\cdot\mathsf{p}\big\} \end{aligned}\right\} \tag{259}$$

Such operators are dimensionless, satisfy (compare (146))

$$[\mathsf{a},\mathsf{b}] = \mathsf{I} \tag{260}$$

and permit one to write

$$\left.\begin{aligned} \mathsf{L}_0 &= \tfrac{\hbar}{2}(\mathsf{b}_1\mathsf{a}_1 + \mathsf{b}_2\mathsf{a}_2) + \tfrac{\hbar}{2}\mathsf{I} \\ \mathsf{L}_1 &= \tfrac{\hbar}{2}(\mathsf{b}_1\mathsf{a}_1 - \mathsf{b}_2\mathsf{a}_2) \\ \mathsf{L}_2 &= \tfrac{\hbar}{2}(\mathsf{b}_1\mathsf{a}_2 + \mathsf{b}_2\mathsf{a}_1) \\ \mathsf{L}_3 &= i\tfrac{\hbar}{2}(\mathsf{b}_1\mathsf{a}_2 - \mathsf{b}_2\mathsf{a}_1) \end{aligned}\right\} \tag{261}$$

If, with Jauch & Rohrlich, we introduce dimensionless observables

$$\left.\begin{aligned} \boldsymbol{\Sigma}_0 &\equiv \quad \mathsf{b}_1\mathsf{a}_1 + \mathsf{b}_2\mathsf{a}_2 \ = \tfrac{2}{\hbar}\mathsf{L}_0 - \mathsf{I} \\ \boldsymbol{\Sigma}_1 &\equiv \quad \mathsf{b}_1\mathsf{a}_1 - \mathsf{b}_2\mathsf{a}_2 \ = \tfrac{2}{\hbar}\mathsf{L}_1 \\ \boldsymbol{\Sigma}_2 &\equiv \quad \mathsf{b}_1\mathsf{a}_2 + \mathsf{b}_2\mathsf{a}_1 \ = \tfrac{2}{\hbar}\mathsf{L}_2 \\ \boldsymbol{\Sigma}_3 &\equiv i(\mathsf{b}_1\mathsf{a}_2 - \mathsf{b}_2\mathsf{a}_1) = \tfrac{2}{\hbar}\mathsf{L}_3 \end{aligned}\right\} \tag{262}$$

then (258) become

$$\begin{aligned} \boldsymbol{\Sigma}_1^2 + \boldsymbol{\Sigma}_2^2 + \boldsymbol{\Sigma}_3^2 &= (\boldsymbol{\Sigma}_0 + \mathsf{I})^2 - \mathsf{I} \\ &= \boldsymbol{\Sigma}_0^2 + 2\boldsymbol{\Sigma}_0 \end{aligned} \tag{263.1}$$

$$[\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2] = 2i\boldsymbol{\Sigma}_3, \quad [\boldsymbol{\Sigma}_2,\boldsymbol{\Sigma}_3] = 2i\boldsymbol{\Sigma}_1, \quad [\boldsymbol{\Sigma}_3,\boldsymbol{\Sigma}_1] = 2i\boldsymbol{\Sigma}_2 \tag{263.2}$$

$$[\boldsymbol{\Sigma}_0,\boldsymbol{\Sigma}_1] = [\boldsymbol{\Sigma}_0,\boldsymbol{\Sigma}_2] = [\boldsymbol{\Sigma}_0,\boldsymbol{\Sigma}_3] = \mathbf{0} \tag{263.3}$$

which agree precisely with equations presented on p. 45 of Jauch & Rohrlich. Apart from a dimensional factor, my operators L_μ and their "Stokes operators" $\boldsymbol{\Sigma}_\mu$ differ only with respect to management of the "zero-point energy term" (which they, who are concerned the oscillatory motion of a particle but with electromagnetic plane waves, are content to abandon). In view of our present interest in the *geometrical design of particulate orbits* it becomes most natural to work with "mechanical Stokes operators" $\mathbf{S}_\mu$ which bear the physical dimension of $(\text{length})^2$:

$$\left.\begin{aligned} \mathbf{S}_0 &\equiv \tfrac{1}{m\omega}\mathsf{L}_0 = \tfrac{\hbar}{2m\omega}\boldsymbol{\Sigma}_0 + \text{insignificant constant} \\ \mathbf{S}_1 &\equiv \tfrac{1}{m\omega}\mathsf{L}_1 = \tfrac{\hbar}{2m\omega}\boldsymbol{\Sigma}_1 \\ \mathbf{S}_2 &\equiv \tfrac{1}{m\omega}\mathsf{L}_2 = \tfrac{\hbar}{2m\omega}\boldsymbol{\Sigma}_2 \\ \mathbf{S}_3 &\equiv \tfrac{1}{m\omega}\mathsf{L}_3 = \tfrac{\hbar}{2m\omega}\boldsymbol{\Sigma}_3 \end{aligned}\right\} \tag{264}$$

It becomes natural to associate "quantum mechanical Stokes parameters" S_μ with quantum states $|\psi)$ and mixtures $\boldsymbol{\rho}$

$$S_\mu = \begin{cases} (\psi|\mathbf{S}_\mu|\psi) & : \quad \text{pure state} \\ \text{trace}(\boldsymbol{\rho}\mathbf{S}_\mu) & : \quad \text{mixed state} \end{cases} \tag{265}$$

and to inquire into the status of the anticipated statement[91]

$$S_0^2 \geqslant S_1^2 + S_2^2 + S_3^2 \tag{266}$$

But before we can address such issues we must confront this prior question: *What can the parameters S_μ possibly mean* in a context where (as emphasized in connection with (249)) the "orbit-tracing pencil" $\big\{\langle \mathsf{x}_1\rangle_t, \langle \mathsf{x}_2\rangle_t\big\}$ hovers at the origin whenever $|\psi)$ is an energy eigenfunction? The question becomes: Why does the pencil hover, and what must we do to get it into motion. . . tracing the ellipses to which the Stokes parameters presumably refer?

Look again, in the interest of expository simplicity, to the one-dimensional oscillator: we have[92]

$$\begin{aligned} \mathsf{H}|n) &= E_n|n) \quad \text{with} \quad \mathsf{H} = \hbar\omega(\mathsf{ba} + \tfrac{1}{2}\mathsf{I}) \\ E_n &= (n + \tfrac{1}{2})\hbar\omega \end{aligned}$$

with

$$|n) = \tfrac{1}{\sqrt{n!}}\,\mathsf{b}^n|0) \tag{267}$$

where the ground state $|0)$ is defined by the conditions $\mathsf{a}|0) = 0$ and $(0|0) = 1$. If $|\psi)_0 = \sum |n)(n|\psi)_0$ then

$$|\psi)_t = e^{-\frac{i}{\hbar}\frac{1}{2}\hbar\omega t}\sum |n)e^{-\frac{i}{\hbar}n\hbar\omega t}(n|\psi)_0 \tag{268}$$

All harmonics of ω are, in the general case, present in the motion of $|\psi)_t$, and present therefore in the motion of

$$\langle \mathsf{A}\rangle_t = \sum_{m,n} e^{i(m-n)\omega t}{}_0(\psi|m)(m|\mathsf{A}|n)(n|\psi)_0 \tag{269}$$

How, in this light, does it happen that harmonics are absent from the motion of $\langle \mathsf{x}\rangle_t$? The answer lies actually close at hand, for it is an implication of (259) that

$$\mathsf{x} = \tfrac{1}{2}\underbrace{\sqrt{\tfrac{2\hbar}{m\omega}}}_{\text{amplitude of classical oscillator with energy } E = \hbar\omega}(\mathsf{b} + \mathsf{a}) \tag{270}$$

[91] Jauch & Rohrlich (who work in a different physical context) assert that the statement is true but "much harder to prove than the corresponding inequality in the classical case," and that equality can hold only if the state is pure. They supply, however, no supporting argument, and cite no reference.

[92] See, for example, QUANTUM MECHANICS (1967), Chapter 2, pp. 48 *et seq.*

and of (267) that

$$\begin{aligned}(m|(\mathsf{b}+\mathsf{a})|n) &= \sqrt{n+1}(m|n+1) + \sqrt{m+1}(m+1|n)\\ &= 0 \text{ unless } m-n=\pm1\end{aligned}$$

The matrix $\mathbb{X} \equiv \|(m|\mathsf{x}|n)\|$ is therefore quite sparce:

$$\mathbb{X} = \tfrac{1}{2}\sqrt{\tfrac{2\hbar}{m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots\\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots\\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots\\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots\\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{271}$$

From the design of $\mathbb{X}$ it follows that

$$\begin{aligned}(\psi|\mathsf{x}|\psi) &= \tfrac{1}{2}\sqrt{\tfrac{2\hbar}{m\omega}}\Big\{\sqrt{1}(\psi_0^*\psi_1+\psi_1^*\psi_0)+\sqrt{2}(\psi_1^*\psi_2+\psi_2^*\psi_1)+\cdots\Big\} \qquad (272)\\ &= 0 \text{ unless } \begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \vdots\end{pmatrix} \begin{array}{l}\text{presents at least one pair of}\\ \text{contiguous non-zero elements}\end{array}\end{aligned}$$

The 0's on the principal diagonal of $\mathbb{X}$ reproduce the statements (compare (250))

$$\langle\mathsf{x}\rangle \equiv (\psi|\mathsf{x}|\psi) = 0 \text{ if } |\psi) \text{ is an energy eigenstate}$$

while the design of (272) accounts for the absence of harmonics in the motion of $\langle\mathsf{x}\rangle_t$: the $e^{\frac{i}{\hbar}(m-n)\omega t}$-factors evident in (269) are annihilated except in cases $m-n=\pm1$. From

$$\mathbb{X}^2 = \tfrac{1}{4}\tfrac{2\hbar}{m\omega}\begin{pmatrix} 1 & 0 & \sqrt{1\cdot2} & 0 & 0 & \cdots\\ 0 & 1+2 & 0 & \sqrt{2\cdot3} & 0 & \cdots\\ \sqrt{1\cdot2} & 0 & 2+3 & 0 & \sqrt{3\cdot4} & \cdots\\ 0 & \sqrt{2\cdot3} & 0 & 3+4 & 0 & \cdots\\ 0 & 0 & \sqrt{3\cdot4} & 0 & 4+5 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{273}$$

it becomes apparent that

$$\begin{aligned}\langle\mathsf{x}^2\rangle_t = \tfrac{1}{4}\tfrac{2\hbar}{m\omega}\Big\{&\sum_0^\infty(2n+1)\psi_n^*\psi_n \qquad (274)\\ &+\sum_0^\infty\sqrt{(n+1)(n+2)}\Big[e^{i2\omega t}\psi_n^*\psi_{n+2}+\text{conjugate}\Big]\Big\}\end{aligned}$$

This result shows in explicit detail how it comes about that

- $\langle \mathsf{x}^2\rangle_t$ possesses a DC component for every $|\psi)$;
- the oscillatory component of $\langle \mathsf{x}^2\rangle_t$—if present—buzzes at the first harmonic of the fundamental frequency ω;
- an oscillatory component will be present if and only if, when parsing the sequence $\{\psi_0, \psi_1, \psi_2, \dots\}$, one encounters $\psi_n\psi_{n+2} \neq 0$ at least once.

From

$$\mathsf{p} = i\sqrt{\tfrac{\hbar m\omega}{2}}(\mathsf{b} - \mathsf{a})$$

one is led to similarly explicit descriptions of $\mathbb{P} \equiv \|(m|\mathsf{p}|n)\|$ and $\mathbb{P}^2$, whence of $\langle \mathsf{p}\rangle_t$ and $\langle \mathsf{p}^2\rangle_t$, but I will not pursue those details. As a check on the accuracy of results thus obtained one has

$$\mathbb{H} = \tfrac{1}{2m}\big\{\mathbb{P}^2 + m^2\omega^2\mathbb{X}^2\big\} = \hbar\omega\begin{pmatrix} 0+\frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 1+\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 2+\frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & 3+\frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

We are in position now to look computationally into credentials of the proposition[93] that

quantum mechanics $\rightarrow$ classical *as the quantum numbers become large*

To that end, let us suppose that the non-zero elements of $\{\psi_0, \psi_1, \psi_2, \dots\}$ are nested in the neighborhood of some large n

$$\big\{0, 0, \dots, 0, 0, \underbrace{\psi_n, \psi_{n+1}, \dots, \psi_{n+k}}_{\text{nest}}, 0, 0, \dots\big\} \tag{275.1}$$

which in the simplest instance entails

$$\tfrac{1}{\sqrt{k+1}}\big\{0, 0, \dots, 0, 0, 1, 1, \dots, 1, 0, 0, \dots\big\} \tag{275.2}$$

Then

$$\begin{aligned}\langle \mathsf{x}\rangle_0 &= \tfrac{1}{2}\sqrt{\tfrac{2\hbar}{m\omega}}\,\tfrac{1}{k+1}\begin{pmatrix}1\\1\\1\\\vdots\\1\\1\end{pmatrix}^{\!\mathsf{T}}\begin{pmatrix} 0 & \sqrt{n+1} & 0 & \cdots & 0 & 0 \\ \sqrt{n+1} & 0 & \sqrt{n+2} & \cdots & 0 & 0 \\ 0 & \sqrt{n+2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{n+k} \\ 0 & 0 & 0 & \cdots & \sqrt{n+k} & 0 \end{pmatrix}\begin{pmatrix}1\\1\\1\\\vdots\\1\\1\end{pmatrix}\\ &= \tfrac{1}{2}\sqrt{\tfrac{2\hbar}{m\omega}}\,\mathbf{2}\,\tfrac{1}{k+1}\sum_{j=1}^{k}\sqrt{n+j}\end{aligned}$$

[93] See, for example, §15–4 in L. E. Ballentine, *Quantum Mechanics* (1990).

But[94]

$$\begin{aligned}\tfrac{1}{k+1}\sum_{j=1}^{k}\sqrt{n+j} &= \sqrt{n}\,\tfrac{1}{k+1}\sum_{j=1}^{k}\Big\{1+\tfrac{1}{2n}j+\cdots\Big\}\\ &= \sqrt{n}\,\tfrac{k}{k+1}+\tfrac{k}{4\sqrt{n}}+\cdots\\ &\sim \sqrt{n} \quad : \quad n \gg k \gg 1\end{aligned}$$

so we have

$$\langle \mathsf{x}\rangle_0 \sim \sqrt{\tfrac{2}{m\omega^2}\hbar\omega n} = \text{amplitude of classical oscillator with energy } E=\hbar\omega n$$

And if we "turn on the clock" the boldface **2** becomes $e^{i\omega t}+e^{-i\omega t}=2\cos\omega t$ giving

$$\langle \mathsf{x}\rangle_t = \langle \mathsf{x}\rangle_0 \cos\omega t \tag{276}$$

So for states $|\psi)$ of the design (275.2) the first moment moves "classically" (oscillates, with the anticipated amplitude). But the point of the discussion emerges only when one looks in this light to the second moment:

States of the nested design (275) render irrelevant all but a $(k+1)\times(k+1)$ submatrix of $\mathbb{X}^2$, which lives far down the principal diagonal, and looks like this:

$$\tfrac{1}{4}\tfrac{2\hbar}{m\omega}\begin{pmatrix} (n+0)+(n+1) & 0 & \sqrt{(n+1)(n+2)} & 0 & \cdots\\ 0 & (n+1)+(n+2) & 0 & \sqrt{(n+2)(n+3)} & \cdots\\ \sqrt{(n+1)(n+2)} & 0 & (n+2)+(n+3) & 0 & \cdots\\ 0 & \sqrt{(n+2)(n+3)} & 0 & (n+3)+(n+4) & \cdots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

The principal diagonal terminates with $(n+k)+(n+k+1)$ and its next-nearest neighbors terminate with $\sqrt{(n+k-1)(n+k)}$. It follows that

$$\langle \mathsf{x}^2\rangle_t = A + B\cos 2\omega t$$

with

$$\begin{aligned}A &= \tfrac{1}{4}\tfrac{2\hbar}{m\omega}\tfrac{1}{k+1}\Big\{(k+1)2n+\sum_{j=0}^{k}(2j+1)\Big\}\\ &= \tfrac{1}{4}\tfrac{2\hbar}{m\omega}2n\big\{1+\tfrac{k+1}{2n}\big\}\end{aligned}$$

$$\begin{aligned}B &= \tfrac{1}{4}\tfrac{2\hbar}{m\omega}2\tfrac{1}{k+1}\sum_{j=2}^{k}\sqrt{(n+j-1)(n+j)}\\ &= \tfrac{1}{4}\tfrac{2\hbar}{m\omega}2\tfrac{1}{k+1}\sqrt{(n-1)n}\sum_{j=2}^{k}\big\{1+\tfrac{1}{2}\big(\tfrac{1}{n-1}+\tfrac{1}{n}\big)j+\cdots\big\}\\ &= \tfrac{1}{4}\tfrac{2\hbar}{m\omega}2n\sqrt{1-\tfrac{1}{n}}\,\tfrac{k-1}{k+1}\Big\{1+\tfrac{1}{4}\big(\tfrac{1}{n-1}+\tfrac{1}{n}\big)(k+2)+\cdots\Big\}\end{aligned}$$

---

[94] We are reminded by *Mathematica* that $\sum_{j=1}^{k}\sqrt{n+j}$ can be expressed in terms of the "generalized Riemann zeta function."

For $n \gg k \gg 0$ we therefore have

$$A \sim B \sim \tfrac{\hbar}{m\omega}n = \tfrac{1}{2}\langle \mathsf{x}\rangle_0^2$$

So for such states

$$\begin{aligned}\langle \mathsf{x}^2\rangle_t &= \tfrac{1}{2}\langle \mathsf{x}\rangle_0^2 + \tfrac{1}{2}\langle \mathsf{x}\rangle_0^2 \cos 2\omega t\\ &= \quad \langle \mathsf{x}\rangle_0^2 \cos^2\omega t\\ &= \langle \mathsf{x}\rangle_t \cdot \langle \mathsf{x}\rangle_t \end{aligned} \tag{276}$$

It is, as I have emphasized elsewhere,[86] the onset of statements of the form

$$\langle \mathsf{A}^p\rangle = \langle \mathsf{A}\rangle^p, \quad \langle \mathsf{AB}\rangle = \langle \mathsf{A}\rangle\langle \mathsf{B}\rangle, \quad \text{etc.}$$

which announces that distributions have become sharply localized (centered moments have vanished) and we have penetrated the world of *classical* physics. Comments of several sorts are now in order.

How robust is the argument which gave (276)? Given the nested design (275), can one (consistently with normalization) assign values *arbitrarily* to the complex numbers ψ_j; writing $\psi_j = r_j e^{i\varphi_j}$, can one place the point $\boldsymbol{r}$ *anywhere* on the surface of the unit $(k+1)$-sphere? Certainly not, for alternating 0's would bring $\langle \mathsf{x}\rangle_t$ to rest at the origin. Do such contrived exceptions perhaps comprise a "set of measure zero"? I suspect not, but...

If one were in possession of moments $\langle \mathsf{x}^p\rangle_0$ of *all* orders then one could reconstruct the numbers r_j, and it would become apparent in high order that the distribution is not exquisitely sharp, and the seeming classical mechanics is a low-order illusion. For $p > k$ the non-zero diagonals in $\mathbb{X}^p$ have moved so far away from the principal diagonal as to become invisible to the $|\psi)$ of (275), with the consequence that $\langle \mathsf{x}^p\rangle_t$ has lost the time-dependence which classically it retains. To phrase the issue another way: if we possessed $\langle \mathsf{x}^p\rangle_t = \langle \mathsf{x}\rangle_t^p$ then the motion of

$$\langle \mathsf{x}\rangle_t \quad \text{and} \quad p \equiv \tfrac{1}{m}\tfrac{d}{dt}\langle \mathsf{x}\rangle_t$$

would present a replication of classical Hamiltonian mechanics—in seeming violation of the uncertainty principle.

The "large quantum number" model sketched above provides no answer to the question: How, in natural fact, would states of the specialized design (275) come spontaneously into being? And it ignores the fact that one cannot, by any standard account of the quantum theory of measurement, "watch" the motion to which $\langle \mathsf{x}\rangle_t$ refers, the relationship of which to the motion of the classical oscillators (which one manifestly *can* watch) remains therefore obscure.[95]

The theory sketched above hinges critically on the double presumption that (*i*) n is very large (formally: $n \to \infty$) and (*ii*) k, though small, is yet not

[95] Discerning critic though he was, Ehrenfest's paper was quite brief, and the point seems not to have concerned him.

too small. For if the "nest" is exquisitely narrow (i.e., if the state is an energy eigenstate, corresponding to the situation $k = 0$) then $\langle \mathsf{x} \rangle$ is immobilized; to launch $\langle \mathsf{x} \rangle$ into motion the state vector $|\psi)_t$—thought of as a "signal"—must have finite bandwidth, and the associated distribution $|(n|\psi)|^2$ must be "course-grained," a *fat* δ-function. This line of commentary places one in position to appreciate the fundamental distinction between the present program (where the objective is to expose the relationship between $\langle \mathsf{x} \rangle_t$ and its classical counterpart $x(t)$) and the more frequently encountered program[96] wherein one compares the static properties of $P_n(x) \equiv |(x|n)|^2$ with those of the function

$$Q_n(x) = \begin{cases} \frac{1}{\pi\sqrt{a^2 - x^2}} & : \quad -a < x < +a, \text{ with } a \equiv \sqrt{\frac{2\hbar}{m\omega}(n + \frac{1}{2})} \\ 0 & : \quad \text{otherwise} \end{cases}$$

$$= \text{fraction of the time } x(t) = a\cos\omega t \text{ spends in the neighborhood } dx \text{ of } x$$

The latter program culminates in figures of a sort which are which are frequently reproduced (see Figure 16) but seldom commented upon; I take this opportunity to draw attention to the following circumstances:

- the program is lopsided/skew insofar as it associates quantum statics with an implication of classical dynamics

- the program—which stands with one toe on a tacit "ergodic hypothesis"—implicitly imputes a classical mechanism to quantum statistics

- the program would have us acknowledge the similarity of curves which are in fact *not* similar; which become similar only after $P_n(x)$ has been subjected to some kind of "local averaging"

$$P_n(x) \rightarrow \tilde{P}_n(x) = \int P_n(\xi) w(x - \xi)\, d\xi$$

which the eye does spontaneously, but of which standard quantum theory provides no account.[97]

- the 2-dimensional analog of Figure 16 is disappointingly uninformative: $Q(x_1, x_2)$ reflects the shape of the "bounding box," but is insensitive to relative phase (so is common to all elliptical orbits inscribed within a given bounding box), while only one product wave function fits within any given bounding box; in Stokes' terminology, the construction has things to say about S_0 and S_1, but nothing about S_2 or S_3.

In short: the line of argument which yielded Figure 16 provides yet further evidence that hints/anticipations of classical mechanics can be detected even deep within the quantum realm, but poses more questions than it resolves.

---

[96] See, for example, Schiff's *Quantum Mechanics* (3rd edition 1968) Fig. 11; D. Griffiths' *Introduction to Quantum Mechanics* (1995) Fig. 2.5; QUANTUM MECHANICS (1967) Chapter 2, pp. 74 *et seq.*

[97] In this connection see J. L. Powel & B. Crasemann, *Quantum Mechanics* (1961), p. 137.

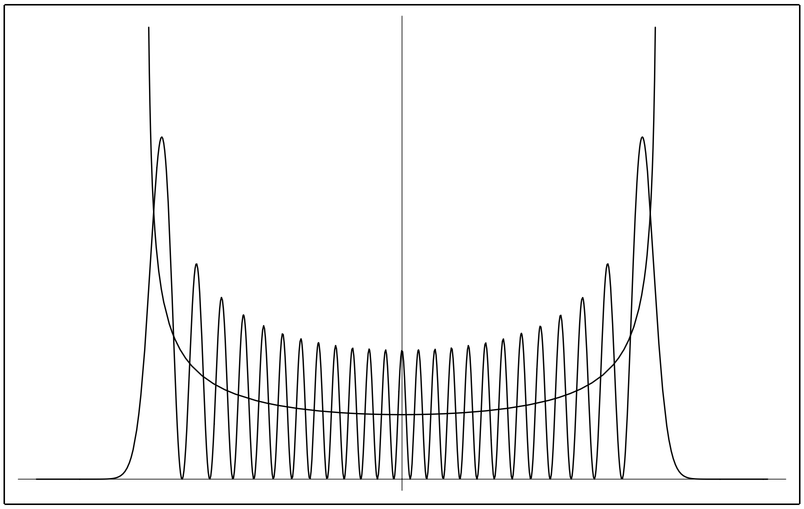

FIGURE 16: *In terms of the dimensionless variable* $y \equiv \sqrt{m\omega/\hbar}x$ *the oscillator eigenfunctions* $(x|n)$ *can be described*

$$h_n(y) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\texttt{HermiteH[n,y]Exp[-}\tfrac{1}{2}\texttt{y}^2\texttt{]}$$

Superimposed in the figure are $P_{24}(x)$ *and the associated* $Q_{24}(y)$.

Textbooks describe also an alternative—and for our purposes more informative—way to relate the quantum mechanics to the classical mechanics of an oscillator. The idea—which according to Schiff was original to Schrödinger himself (1926)—is to assign $\psi(x,0)$ a design consistent with the statement

$$\begin{aligned}|\psi(x,0)|^2 &= \tfrac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{1}{2\epsilon}(x-a)^2}\\ &= \text{normalized Gaussian}\end{aligned}$$

and then to watch the motion of $|\psi(x,t)|^2$. This is standardly accomplished either by writing[98]

$$\begin{aligned}\psi(x,t) &= \sum c_n e^{-i(n+\frac{1}{2})\omega t}\psi_n(x)\\ c_n &= \int \psi_n(x')\psi(x',0)\,dx'\end{aligned}$$

or by writing[99]

$$\psi(x,t) = \int K(x,t;x',0)\psi(x',0)\,dx'$$

[98] See, for example, Schiff pp. 74–76 for the details.
[99] See QUANTUM MECHANICS (1967) Chapter 2, p. 93.

with

$$\begin{aligned} K(x,t;x',0) &= \sum \psi_n(x)\psi_n(x')e^{-i(n+\frac{1}{2})\omega t} \\ &= \sqrt{\tfrac{m\omega}{2\pi i\hbar \sin\omega t}}\,\exp\left\{-\tfrac{m\omega}{2i\hbar\sin\omega t}\big[(x^2+x'^2)\cos\omega t - 2xx'\big]\right\} \end{aligned}$$

I proceed here in language appropriate a third method (which, however, I will not attempt to develop in detail). The time-dependent Schrödinger equation of an oscillator reads

$$-\varphi_{yy} + y^2\varphi = 2i\varphi_u \tag{277}$$

when written in dimensionless variables $y = \sqrt{m\omega/\hbar}\,x$ and $u = \omega t$. To develop a general theory of "dispersionless oscillator wavepackets" one would proceed from the weakest assumption

$$\varphi(y,u) = F(y - b\cos u)e^{if(y,u)} \tag{278}$$

sufficient to insure that

$$|\varphi(y,u)|^2 = F^2(y - b\cos u)$$

oscillates rigidly/classically (with amplitude b, angular frequency ω); the plan might be to specify $F(\bullet)$ arbitrarily, then require, as a condition on $f(y,u)$, that the φ of (278) satisfies (277). Here I must be content to observe that Schiff takes $F = \sqrt{\text{displaced Gaussian}}$, but not just any Gaussian: to avoid notational clutter he takes

$$\begin{aligned} \varphi(y,0) &= h_0(y-b) \quad : \quad \text{displaced ground state} \\ &= \left(\tfrac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}(y-b)^2} \end{aligned}$$

and computes (as *Mathematica* confirms) $c_n = b^n e^{-\frac{1}{4}b^2}/\sqrt{2^n n!}$, which is readily shown to entail

$$\varphi(y,u) = Fe^{if} \quad \text{with} \quad \begin{cases} F = h_0(y - b\cos u) \\ f(y,u) = -\frac{1}{2}u - by\sin u + \frac{1}{4}b^2\sin 2u \end{cases}$$

which (by quick calculation) does in fact satisfy the Schrödinger equation (277). The resulting probability density is the "sloshing Gaussian" shown in Figure 17. We note that the second harmonic is present in f, but absent from F. And that $\varphi(y,u) \to h_0(y)\cdot e^{-\frac{1}{2}u}$ as $b \downarrow 0$: the ground state "buzzes with zero-point frequency" and $|\varphi(y,u)|^2$ has become an immobilized Gaussian. As amplitude b increases, progressively larger quantum numbers n become prominent in the development

$$\varphi(y,u) = \sum c_n e^{-i(n+\frac{1}{2})u}h_n(y) = h_0(y - b\cos u)\cdot e^{i(\text{phase function})}$$

Schiff, in that connection, observes that (in Stirling approximation)

$$\begin{aligned} \log c_n &= n\big(\log b - \tfrac{1}{2}\log 2\big) - \tfrac{1}{2}\log n! - \tfrac{1}{4}b^2 \\ &\approx n\big(\log b - \tfrac{1}{2}\log 2\big) - \tfrac{1}{2}\big\{\log\sqrt{2\pi} + (n+\tfrac{1}{2})\log n - n\big\} - \tfrac{1}{4}b^2 \end{aligned}$$

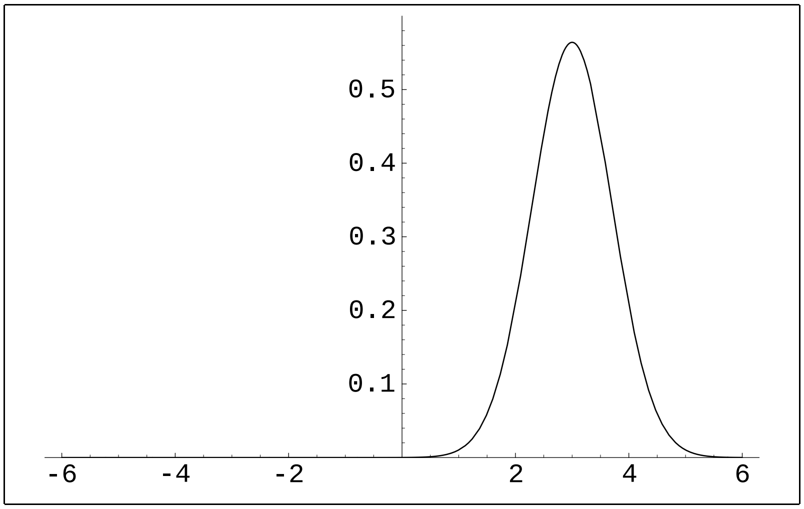

FIGURE 17: *One frame from the animated graphic generated by*

```
h[n_,y_]:=1/√(2^n n! √π) HermiteH[n,y] Exp[-½y²]
g[y_,u_]:=h[0,y-3Cos[u]]
Animate[Plot[g[y,u], {y,-6,+6}, PlotRange->{0,0.6}],
                                              {u,0,2π}]
```

and set into sloshing motion by opening the `Cell` *menu and clicking on* `Animate Selected Graphics`*.*

becomes maximal at

$$n_{\max}(b) = \tfrac{1}{2}b^2\big(1 - \tfrac{1}{b^2} + \cdots\big) \approx \tfrac{1}{2}(b^2 - 1) \tag{279.1}$$
$$b \equiv \sqrt{\tfrac{m\omega}{\hbar}}\cdot(\text{literal amplitude } a)$$

This—gratifyingly if not surprisingly—is precisely the n-value which follows from writing $E_n = \hbar\omega(n + \frac{1}{2}) = \frac{1}{2}m\omega^2 a^2$. Expansion about $n_{\max}$ gives

$$c_n \approx \big(\tfrac{1}{\pi b^2}\big)^{\frac{1}{4}} \exp\Big\{-\tfrac{1}{2}\Big[\frac{n - n_{\max}}{b}\Big]^2\Big\} \tag{279.2}$$

These results are illustrated in Figures 17 and 18.

It is clear (especially from the latter figure) that the preceding discussion has much in common with that which proceeded from (275). The general lesson, once again, it that to launch first moments into motion the quantum state must be of the form

$$|\psi) = |\text{eigenfunction}) + \sum|\text{nest of neighboring eigenfunctions})$$

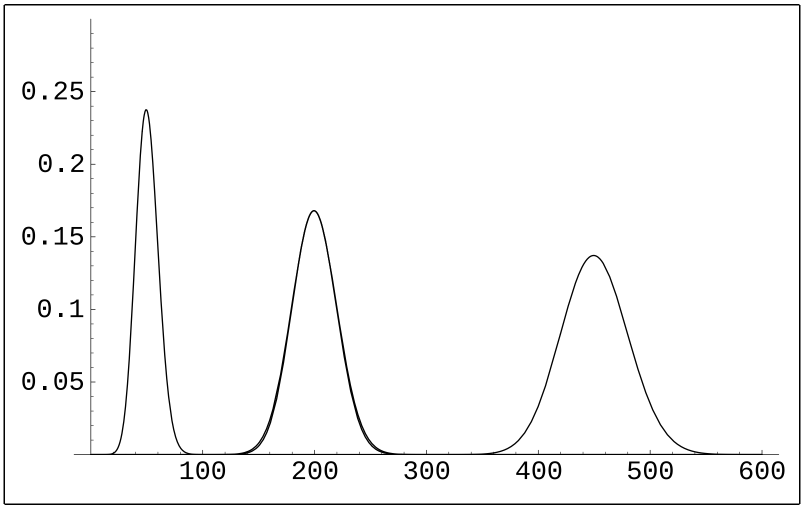

FIGURE 18: *Graphs of*

$$c_n(b) = b^n e^{-\frac{1}{4}b^2}/\sqrt{2^n n!}$$

in the cases $b = 10, 20, 30$. *The peaks become shorter but fatter as* b *increases. The peaks are placed at*

$$n_{\max}(b) \approx \tfrac{1}{2}(b^2 - 1) = \begin{cases} 49.5 \\ 199.5 \\ 449.5 \end{cases}$$

respectively, and have

$$\textit{altitudes} \approx \left(\tfrac{1}{\pi b^2}\right)^{\frac{1}{4}} = \begin{cases} 0.2375 \\ 0.1680 \\ 0.1371 \end{cases}$$

The central peak is actually a superposition of the exact curve and its approximant (279.2); the fit is already spectacular at $b = 20$, *and gets rapidly even better as the dimensionless amplitude* b *increases.*

and that to successfully mimic classical physics the mean quantum number must be large. I elaborate upon the latter remark as it pertains to the "sloshing groundstate:" From the circumstance that

$$\begin{aligned} \psi(x,t) &= \psi_0(x - a\cos\omega t)\cdot e^{i(\text{phase function})} \\ &\Downarrow \\ P(x,t) &= |\psi_0(x - a\cos\omega t)|^2 \text{ moves "rigidly"} \end{aligned}$$

it follows that the *centered* moments of all orders are *constant* in time. In particular, one has

$$\langle (\mathsf{x} - \langle \mathsf{x}\rangle_t)^2 \rangle_t = \langle \mathsf{x}^2\rangle_t - \langle \mathsf{x}\rangle_t^2 = \text{constant}$$

From the specific Gaussian design of

$$P(x,t) = \sqrt{\tfrac{m\omega}{\pi\hbar}}\exp\left\{\tfrac{m\omega}{\hbar}(x - a\cos\omega t)^2\right\} \tag{280}$$

it follows that “constant”$= \hbar/2m\omega$. So the “sloshing groundstate” provides

$$\langle \mathsf{x}^2\rangle_t = \langle \mathsf{x}\rangle_t\cdot\langle \mathsf{x}\rangle_t + \tfrac{1}{2}(\text{zero-point amplitude})^2 \tag{281}$$

and we penetrate the “classical realm” when circumstances ($E \gg \hbar\omega$, or equivalently: $a \gg \sqrt{\hbar/m\omega}\,$) permit neglect of the additive term.

All of which, though phrased in reference to the one-dimensional oscillator, pertains *mutatis mutandis* (and with very little *mutatis*) to the quantum physics of two-dimensional oscillators, and in particular to the isotropic oscillator. We retain the insight that if the first moments are to be launched into (necessarily elliptical) motion the state must present a superimposed “nest” of eigenstates, and in the “sloshing groundstate” (see the following figure)

$$\begin{aligned}\varphi(y_1,y_2,u;b_1,b_2,\delta) =&\varphi(y_1,u;b_1)\cdot\varphi(y_2,u+\delta;b_2) \\ \varphi(y,u;b) &= h_0(y-b\cos u)e^{i\left\{-\frac{1}{2}u - by\sin u + \frac{1}{4}b^2\sin 2u\right\}} \\ h_0(y) &= \left(\tfrac{1}{\pi}\right)^{\frac{1}{4}}e^{-\frac{1}{2}y^2}\end{aligned} \tag{282}$$

possess an analytically tractable (yet representative?) instance of such a state. We know from first principles that the “mechanical Stokes operators” S_μ introduced at (264) commute with the oscillator Hamiltonian $\mathsf{H} = 2m\omega^2\mathsf{S}_0$, and therefore that the numbers $S_\mu \equiv (\psi|\mathsf{S}_\mu|\psi)$ remain constant whenever $|\psi)$ moves as stipulated by the Schrödinger equation. We find ourselves in position now (at last!) to inquire in some detail into the question which motivated this entire discussion: What do the numbers S_μ have to say about the design of the “quantum orbit” associated with the state $|\psi)$?

The operators a and b acquire the representations

$$\begin{aligned}\mathsf{a} &= \tfrac{1}{\sqrt{2}}\big(y + \tfrac{\partial}{\partial y}\big) \\ \mathsf{b} &= \tfrac{1}{\sqrt{2}}\big(y - \tfrac{\partial}{\partial y}\big)\end{aligned}$$

in terms of which (262) become

$$\begin{aligned}\mathbf{\Sigma}_0 &= \ \tfrac{1}{2}\big\{\big(y-\tfrac{\partial}{\partial y}\big)_1\big(y+\tfrac{\partial}{\partial y}\big)_1 + \big(y-\tfrac{\partial}{\partial y}\big)_2\big(y+\tfrac{\partial}{\partial y}\big)_2\big\} \\ \mathbf{\Sigma}_1 &= \ \tfrac{1}{2}\big\{\big(y-\tfrac{\partial}{\partial y}\big)_1\big(y+\tfrac{\partial}{\partial y}\big)_1 - \big(y-\tfrac{\partial}{\partial y}\big)_2\big(y+\tfrac{\partial}{\partial y}\big)_2\big\} \\ \mathbf{\Sigma}_2 &= \ \tfrac{1}{2}\big\{\big(y-\tfrac{\partial}{\partial y}\big)_1\big(y+\tfrac{\partial}{\partial y}\big)_2 + \big(y-\tfrac{\partial}{\partial y}\big)_2\big(y+\tfrac{\partial}{\partial y}\big)_1\big\} \\ \mathbf{\Sigma}_2 &= i\tfrac{1}{2}\big\{\big(y-\tfrac{\partial}{\partial y}\big)_1\big(y+\tfrac{\partial}{\partial y}\big)_2 - \big(y-\tfrac{\partial}{\partial y}\big)_2\big(y+\tfrac{\partial}{\partial y}\big)_1\big\}\end{aligned}$$

Equivalent, but computationally more useful, are the statements

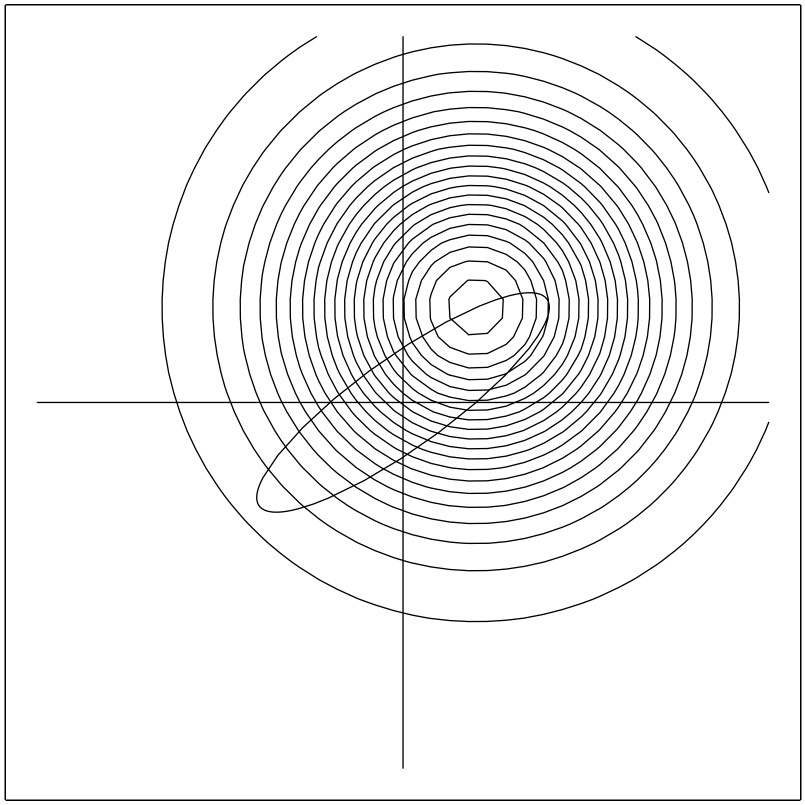

FIGURE 19: *Representation of the isotropic analog of a "sloshing groundstate." A displaced copy of the groundstate moves rigidly/ classically along an elliptical orbit.*

$$\left.\begin{aligned}\mathbf{\Sigma}_0 &= \tfrac{1}{2}\big\{-\partial_1^2+y_1^2\big\}+\tfrac{1}{2}\big\{-\partial_2^2+y_2^2\big\}-1\\ \mathbf{\Sigma}_1 &= \tfrac{1}{2}\big\{-\partial_1^2+y_1^2\big\}-\tfrac{1}{2}\big\{-\partial_2^2+y_2^2\big\}\\ \mathbf{\Sigma}_2 &= \quad y_1y_2-\partial_1\partial_2\\ \mathbf{\Sigma}_3 &= i\big\{y_1\partial_2-y_2\partial_1\big\}\end{aligned}\right\}\tag{283}$$

In the one-dimensional case *Mathematica*, given $\varphi=\varphi(y,u;b)$, supplies

$$\begin{aligned}\langle y\ \rangle &= +b\cos u &\qquad \langle -i\partial\rangle &= -b\sin u\\ \langle y^2\rangle &= \tfrac{1}{2}+b^2\cos^2 u &\qquad \langle -\partial^2\rangle &= \tfrac{1}{2}+b^2\sin^2 u\end{aligned}$$

whence $\langle -\partial^2+y^2\rangle = 1+b^2$. Returning with this information to (283), and taking the wave function to be the sloshing groundstate (282), we obtain

$$
\begin{aligned}
\Sigma_0 &\equiv (\varphi|\boldsymbol{\Sigma}_0|\varphi) = \tfrac{1}{2}(1+b_1^2) + \tfrac{1}{2}(1+b_2^2) - 1 \\
&= \tfrac{1}{2}(b_1^2 + b_2^2) && (284.0)\\
\Sigma_1 &\equiv (\varphi|\boldsymbol{\Sigma}_1|\varphi) = \tfrac{1}{2}(b_1^2 - b_2^2) && (284.1)\\
\Sigma_2 &\equiv (\varphi|\boldsymbol{\Sigma}_2|\varphi) = b_1 b_2\big\{\cos u \cos(u+\delta) + \sin u \sin(u+\delta)\big\} \\
&= b_1 b_2 \cos\delta && (284.2)\\
\Sigma_3 &\equiv (\varphi|\boldsymbol{\Sigma}_2|\varphi) = b_1 b_2 \sin\delta && (284.3)
\end{aligned}
$$

So if we introduce *Stokes operators* by means of the modified definitions[100]

$$
\mathsf{S}_\mu \equiv \tfrac{2\hbar}{m\omega}\boldsymbol{\Sigma}_\mu \quad : \quad \text{physical dimension of (length)}^2 \tag{285}
$$

we have

$$
\left.
\begin{aligned}
(\varphi|\mathsf{S}_0|\varphi) &= a_1^2 + a_2^2 \\
(\varphi|\mathsf{S}_1|\varphi) &= a_1^2 - a_2^2 \\
(\varphi|\mathsf{S}_2|\varphi) &= 2a_1 a_2 \cos\delta \\
(\varphi|\mathsf{S}_3|\varphi) &= 2a_1 a_2 \sin\delta
\end{aligned}
\right\} \tag{286}
$$

But these are precisely the Stokes parameters which in classical theory we would associate with the ellipse traced

$$
\begin{aligned}
x_1(t) &= a_1 \cos(\omega t) \\
x_2(t) &= a_2 \cos(\omega t + \delta)
\end{aligned}
$$

by the center of the sloshing groundstate. Though any other result would have been perplexing, it remains nonetheless remarkable that we have been able to express those numbers as *expectation values* (and thus to realize—at least in this special case—an objective stated on p. 111). Manifestly

$$
(\varphi|\mathsf{S}_0|\varphi)^2 - (\varphi|\mathsf{S}_1|\varphi)^2 - (\varphi|\mathsf{S}_2|\varphi)^2 - (\varphi|\mathsf{S}_2|\varphi)^2 = 0 \tag{287}
$$

and we can on this basis assert that the sloshing groundstate is (in language borrowed from optics) "100% polarized:"

$$
\frac{\sqrt{(\varphi|\mathsf{S}_1|\varphi)^2 + (\varphi|\mathsf{S}_2|\varphi)^2 + (\varphi|\mathsf{S}_2|\varphi)^2}}{(\varphi|\mathsf{S}_0|\varphi)} = 1 \tag{288}
$$

How robust are the results exposed by the sloshing groundstate? It is to gain insight into that question that we look now to the expectation values of

---

[100] Note that the operators S_μ differ not only notationally but also in small details from the operators $\mathbf{S}_\mu$ introduced at (264). We are in position now to appreciate the wisdom implicit in Jauch & Rohrlich's management of the "zero-point energy term."

S_μ in cases where the state is an energy eigenstate: $|n_1,n_2)e^{-\frac{i}{\hbar}(n_1+n_2+1)\omega t}$. We work from (262) with the aid of

$$\begin{aligned}\mathsf{a}_1|n_1,n_2) &= \sqrt{n_1}\,|n_1-1,n_2), & \mathsf{a}_2|n_1,n_2) &= \sqrt{n_2}\,|n_1,n_2-1)\\ \mathsf{b}_1|n_1,n_2) &= \sqrt{n_1+1}\,|n_1+1,n_2), & \mathsf{b}_2|n_1,n_2) &= \sqrt{n_2+1}\,|n_1,n_2+1)\end{aligned}$$

and (to avoid the clutter of temporal factors which in the end disappear) agree to set $t=0$. Quick calculation gives

$$\left.\begin{aligned}S_0 &= \tfrac{2\hbar}{m\omega}(n_1,n_2|\boldsymbol{\Sigma}_0|n_1,n_2) = \tfrac{2\hbar}{m\omega}(n_1+n_2)\\ S_1 &= \tfrac{2\hbar}{m\omega}(n_1,n_2|\boldsymbol{\Sigma}_1|n_1,n_2) = \tfrac{2\hbar}{m\omega}(n_1-n_2)\\ S_2 &= \tfrac{2\hbar}{m\omega}(n_1,n_2|\boldsymbol{\Sigma}_2|n_1,n_2) = 0\\ S_3 &= \tfrac{2\hbar}{m\omega}(n_1,n_2|\boldsymbol{\Sigma}_3|n_1,n_2) = 0\end{aligned}\right\} \tag{289}$$

From $S_0^2 \geqslant S_1^2 + 0^2 + 0^2$ we discover that oscillator eigenstates are, in general, only *partially* polarized, with

$$\text{“degree of polarization” } P = \frac{\sqrt{(n_1+n_2)^2 - 4n_1n_2}}{n_1+n_2} = \frac{S_1}{S_0} \tag{290}$$

and become 100% polarized if and only if $n_1n_2 = 0$. We are brought thus back to a point first remarked in connection with Figure 16: the data written into specification of an eigenstate $|n_1,n_2)$ is sufficient to set the shape of the bounding box, but not to distinguish one from another of the ellipses thus circumscribed (see Figure 20). And since the first moments are immobilized there is, in fact, no well-defined quantum ellipse to be distinguished: all are latent. We are in position now to appreciate that when Jauch & Rohrlich, with mixed states in mind, assert[91] that

$$100\%\text{ polarization} \Rightarrow \text{system in a “pure” state}$$

they do not intend the converse (which we have just seen to be generally false).

Enlarging now upon (289), we have

$$(m_1,m_2|\boldsymbol{\Sigma}_0|n_1,n_2) = (n_1+n_2)(m_1,m_2|n_1,n_2) \tag{291.0}$$
$$(m_1,m_2|\boldsymbol{\Sigma}_1|n_1,n_2) = (n_1-n_2)(m_1,m_2|n_1,n_2) \tag{291.1}$$
$$\begin{aligned}(m_1,m_2|\boldsymbol{\Sigma}_2|n_1,n_2) = &\sqrt{(n_1+1)n_2}\,(m_1,m_2|n_1+1,n_2-1)\\ &+\sqrt{n_1(n_2+1)}\,(m_1,m_2|n_1-1,n_2+1)\end{aligned} \tag{291.2}$$
$$\begin{aligned}(m_1,m_2|\boldsymbol{\Sigma}_3|n_1,n_2) = i\big\{&\sqrt{(n_1+1)n_2}\,(m_1,m_2|n_1+1,n_2-1)\\ &-\sqrt{n_1(n_2+1)}\,(m_1,m_2|n_1-1,n_2+1)\big\}\end{aligned} \tag{291.3}$$

The pair of equations describe *diagonal* matrices, while the second pair describe matrices which have non-zero elements only on the *nearest-neighbor* of the (doubly-indexed) diagonal. Writing

$$|\psi) = \sum_{n_1,n_2} |n_1,n_2)(n_1,n_2|\psi)$$

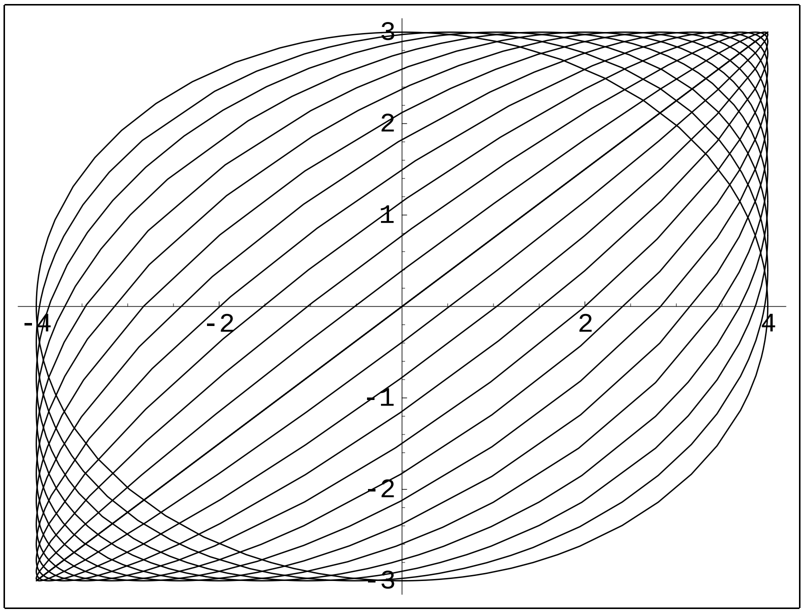

FIGURE 20: *Classical orbits "latent" in the eigenstate* $|n_1, n_2)$. *One has*

$$\begin{aligned} U &\equiv \text{energy} - \text{zero-point} \\ &= \hbar\omega(n_1 + n_2) = \tfrac{1}{2}m\omega^2 S_0 \end{aligned}$$

but—borrowing from §10 *a notion more natural to optics—can also introduce an*

$$\begin{aligned} S &\equiv \text{“internal entropy”} \\ &= -\log\Big\{\big[\tfrac{1}{2}(1+P)\big]^{\frac{1}{2}(1+P)}\big[\tfrac{1}{2}(1-P)\big]^{\frac{1}{2}(1-P)}\Big\} \end{aligned}$$

Specification of U *and* S *is equivalent to specification of the quantum numbers* n_1 *and* n_2. *I will discuss on another occasion what insight might thus be gained.*[101]

and using (291) to compute $(\psi|\mathbf{\Sigma}_\mu|\psi)$, we are led the realization that just as the components of $|\psi)$ must be "nested" to set the first moments $(\psi|\mathsf{x}_1|\psi)$ and $(\psi|\mathsf{x}_2|\psi)$ into (elliptical) motion, so must they be nested to lend non-zero values to S_2 and S_3. And (setting aside the trivial case $n_1 n_2 = 0$) so, finally, must they be nested if we are to achieve the 100% polarization charactistic of the *classical* motion of isotropic oscillators.

20. Transform properties of Stokes parameters. Size and figure (eccentricity) are intrinsic properties of an ellipse, but position and orientation are relative—relative (let us say) to a Cartesian frame which has been erected at (let us say)

[101] In the meantime, see §13 in "Gaussian wavepackets" (1998).

the center of the ellipse in question. It follows that the values assumed by the Stokes parameters descriptive of the ellipse are, in some respects, contingent upon prior specification of the reference frame. Specifically, frame-rotation (see the following figure) entails

$$\begin{aligned}
\mathfrak{X}_1^2 + \mathfrak{X}_2^2 = \mathfrak{X}_1'^{\,2} + \mathfrak{X}_2'^{\,2} \quad &: \quad \text{size-invariance, the upshot of Figure 2}\\
\chi = \chi' \quad &: \quad \text{figure-invariance}\\
\psi = \psi' + \theta &
\end{aligned}$$

from which, working from (20), we obtain

$$\left.\begin{aligned}
S_0 &= S_0'\\
S_1 &= S_1' \cos 2\theta - S_2' \sin 2\theta\\
S_2 &= S_1' \sin 2\theta + S_2' \cos 2\theta\\
S_3 &= S_3'
\end{aligned}\right\} \tag{292}$$

In short (see again the lower portion of Figure 3): rotation of the reference frame induces rotation (through the doubled angle) about the 3-axis in Stokes space.

If one had not geometry but physics—classical oscillators or lightbeams—in mind when one drew the ellipse in Figure 21 then it would be natural to write

$$\begin{aligned}
\mathfrak{X}_1 \cos(\omega t + \delta_1) &= \mathfrak{X}_1' \cos(\omega t) \cdot \cos\theta - \mathfrak{X}_2' \cos(\omega t + \delta') \cdot \sin\theta\\
\mathfrak{X}_2 \cos(\omega t + \delta_2) &= \mathfrak{X}_1' \cos(\omega t) \cdot \sin\theta + \mathfrak{X}_2' \cos(\omega t + \delta') \cdot \cos\theta
\end{aligned}$$

and to develop the implied description of $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\delta \equiv \delta_2 - \delta_1$ in terms of $\mathfrak{X}_1'$, $\mathfrak{X}_2'$ and δ'. This was, in fact, done already in §2, where we were led to equations of disappointing complexity—equations which at (32) led us to our first anticipation of (292), and which, by reversal of our former procedure, might now be recovered directly from (292). It is in that context that I consider now this question: What would be the state of affairs if we had *quantum* oscillators in mind?

The energy eigenvalues of an isotropic oscillator can be described

$$E_n = (n+1)\hbar\omega \quad \text{with} \quad n = n_1 + n_2$$

and are $(n+1)$-fold *degenerate*, the associated eigenstates being

$$|n,0),\ |n-1,1),\ |n-2,2), \ldots,\ |2,n-2),\ |1,n-1),\ |0,n)$$

The dimensionless space representation of the groundstate can, as previously remarked, be described

$$h_{0,0}(y_1,y_2) \equiv (y_1,y_2|0,0) = \left(\tfrac{1}{\pi}\right)^{\frac{2}{4}} e^{-\frac{1}{2}(y_1^2+y_2^2)}$$

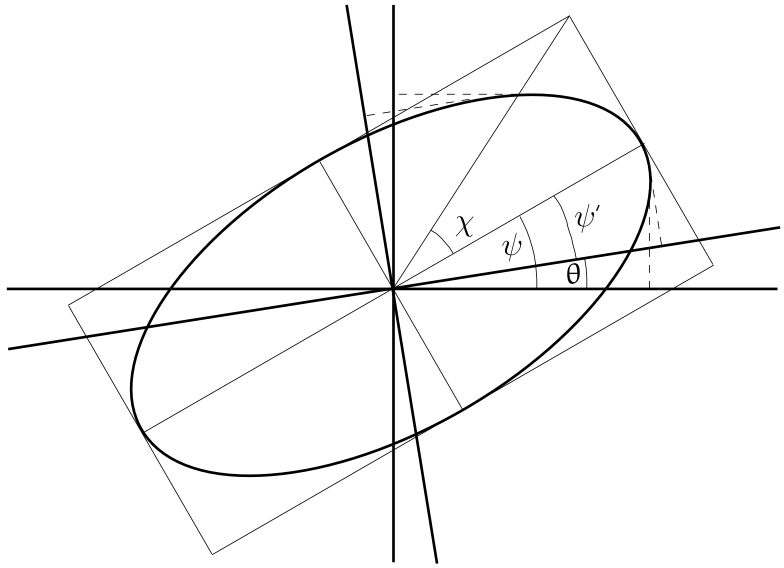

FIGURE 21: *Modification of the upper portion of Figure 3, into which has been introduced a rotated copy of the original Cartesian frame:*

$$\begin{aligned} x_1 &= x_1' \cos\theta - x_2' \sin\theta \\ x_2 &= x_1' \sin\theta + x_2' \cos\theta \end{aligned}$$

and wavefunctions representative of the excited states become

$$\begin{aligned} h_{n_1,n_2}(y_1,y_2) &= (y_1,y_2|\tfrac{1}{\sqrt{n_1!n_2!}}\,\mathsf{b}_1^{n_1}\mathsf{b}_2^{n_2}|0,0) \\ &= \tfrac{1}{\sqrt{n_1!n_2!}}\big[\tfrac{1}{\sqrt{2}}\big(y_1 - \tfrac{\partial}{\partial y_1}\big)\big]^{n_1}\big[\tfrac{1}{\sqrt{2}}\big(y_2 - \tfrac{\partial}{\partial y_2}\big)\big]^{n_2} h_{0,0}(y_1,y_2) \end{aligned}$$

Rotation

$$\begin{aligned} y_1 &= y_1' \cos\theta - y_2' \sin\theta \\ y_2 &= y_1' \sin\theta + y_2' \cos\theta \end{aligned} \quad \text{induces} \quad \begin{aligned} \partial_1 &= \partial_1' \cos\theta - \partial_2' \sin\theta \\ \partial_2 &= \partial_1' \sin\theta + \partial_2' \cos\theta \end{aligned}$$

whence

$$\left.\begin{aligned} \mathsf{b}_1 &= \mathsf{b}_1' \cos\theta - \mathsf{b}_2' \sin\theta \\ \mathsf{b}_2 &= \mathsf{b}_1' \sin\theta + \mathsf{b}_2' \cos\theta \end{aligned}\right\} \tag{293}$$

We are led thus to statements of the form

$$\begin{aligned} &h_{1,0}(y_1' \cos\theta - y_2' \sin\theta, y_1' \sin\theta + y_2' \cos\theta) \\ &\qquad = \cos\theta \cdot h_{1,0}(y_1', y_2') - \sin\theta \cdot h_{0,1}(y_1', y_2') \\ &h_{0,1}(y_1' \cos\theta - y_2' \sin\theta, y_1' \sin\theta + y_2' \cos\theta) \\ &\qquad = \sin\theta \cdot h_{1,0}(y_1', y_2') + \cos\theta \cdot h_{0,1}(y_1', y_2') \end{aligned}$$

which can be notated

$$\begin{pmatrix} h_{1,0}(\mathbf{y}) \\ h_{0,1}(\mathbf{y}) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\mathbb{R}(\theta;1)} \begin{pmatrix} h_{1,0}(\mathbf{y}') \\ h_{0,1}(\mathbf{y}') \end{pmatrix} \tag{294.1}$$

and in next higher order becomes

$$\begin{pmatrix} h_{2,0}(\mathbf{y}) \\ h_{1,1}(\mathbf{y}) \\ h_{0,2}(\mathbf{y}) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos^2\theta & -\sqrt{2}\sin\theta\cos\theta & \sin^2\theta \\ \sqrt{2}\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & -\sqrt{2}\sin\theta\cos\theta \\ \sin^2\theta & \sqrt{2}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}}_{\mathbb{R}(\theta;2)} \begin{pmatrix} h_{2,0}(\mathbf{y}') \\ h_{1,1}(\mathbf{y}') \\ h_{0,2}(\mathbf{y}') \end{pmatrix} \tag{294.2}$$

Equations (294) state allegations which *Mathematica* has explicitly confirmed. *Mathematica* reports also that $\mathbb{R}(\theta;2)$ shares these properties with $\mathbb{R}(\theta;1)$:

- $\mathbb{R}(\theta;2)$ is a rotation matrix: $\mathbb{R}^{\mathsf{T}}(\theta;2)\mathbb{R}(\theta;2) = \mathbb{I}$;
- $\mathbb{R}(\theta_1;2)\mathbb{R}(\theta_2;2) = \mathbb{R}(\theta_1+\theta_2;2)$; the matrices in question provide a 3×3 representation of $O(2)$.

From

$$\begin{aligned} \tfrac{1}{\sqrt{(n-j)!j!}}\,\mathsf{b}_1^{n-j}\,\mathsf{b}_2^{j} &= \tfrac{1}{\sqrt{(n-j)!j!}}(\mathsf{b}_1'\cos\theta - \mathsf{b}_2'\sin\theta)^{n-j}(\mathsf{b}_1'\sin\theta + \mathsf{b}_2'\cos\theta)^{j} \\ &= \tfrac{1}{\sqrt{(n-j)!j!}}\sum_{p=0}^{n-j}\tbinom{n-j}{p}(\mathsf{b}_1'\cos\theta)^p(-\mathsf{b}_2'\sin\theta)^{n-j-p} \\ &\qquad\cdot\sum_{q=0}^{j}\tbinom{j}{q}(\mathsf{b}_1'\sin\theta)^q(\mathsf{b}_2'\cos\theta)^{j-q} \\ &= \sum_{k=0}^{n}\underbrace{R^j{}_k(\theta;n)}_{\text{elements of the } (n+1)\times(n+1) \text{ matrix } \mathbb{R}(\theta;n)}\tfrac{1}{\sqrt{(n-k)!k!}}\,\mathsf{b}_1'^{\,n-k}\,\mathsf{b}_2'^{\,k} \end{aligned} \tag{295}$$

we learn how to ascend to arbitrary order, but detailed description of the matrix elements $R^i{}_j$ involves a degree of tedium into which I am not motivated to enter; the exercise would, in effect, reproduce the representation theory of $O(2)$. I return instead to the question which motivated this discussion.

Let $|\psi)$ refer to the state of an isotropic oscillator, and let $|\psi') = \mathsf{R}(\theta)|\psi)$ be a rotated copy of that state. The question of immediate interest is this: How do the Stokes parameters $(\psi'|\mathbf{\Sigma}_\mu|\psi')$ of $|\psi')$ relate to those of $|\psi)$? We are in position to pursue two distinct approaches to the problem. We might write

$$\begin{aligned} |\psi) &= \sum_n |\psi_n) \\ &\quad |\psi_n) \equiv \sum_{p=0}^{n} |n-p,p)(n-p,p|\psi) \in n^{\text{th}} \text{ energy eigenspace} \end{aligned}$$

and exploit recently-acquired information about how $\mathsf{R}(\theta)$ acts within each such eigenspace. This, however, would be the long way around the bush; it would be much simpler and more informative to write

$$(\psi'|\boldsymbol{\Sigma}_\mu|\psi') = (\psi|\mathsf{R}^{\mathsf{T}}(\theta)\boldsymbol{\Sigma}_\mu\mathsf{R}(\theta)|\psi) = (\psi|\boldsymbol{\Sigma}'_\mu|\psi)$$

and to study the relationship of $\boldsymbol{\Sigma}'_\mu$ to $\boldsymbol{\Sigma}_\mu$; this is a relatively simple matter, and can be pursued independently of any reference to the specific structure of the state vector $|\psi)$. Bringing (293) and their adjoints to (262), we obtain

$$\begin{aligned}\boldsymbol{\Sigma}_0 = &\quad(\mathsf{b}'_1\cos\theta - \mathsf{b}'_2\sin\theta)(\mathsf{a}'_1\cos\theta - \mathsf{a}'_2\sin\theta)\\ &+(\mathsf{b}'_1\sin\theta + \mathsf{b}'_2\cos\theta)(\mathsf{a}'_1\sin\theta + \mathsf{a}'_2\cos\theta)\\ &= \mathsf{b}'_1\mathsf{a}'_1 + \mathsf{b}'_2\mathsf{a}'_2\\ &= \boldsymbol{\Sigma}'_0\end{aligned} \tag{296.0}$$

$$\begin{aligned}\boldsymbol{\Sigma}_1 = &\quad(\mathsf{b}'_1\cos\theta - \mathsf{b}'_2\sin\theta)(\mathsf{a}'_1\cos\theta - \mathsf{a}'_2\sin\theta)\\ &-(\mathsf{b}'_1\sin\theta + \mathsf{b}'_2\cos\theta)(\mathsf{a}'_1\sin\theta + \mathsf{a}'_2\cos\theta)\\ &= (\mathsf{b}'_1\mathsf{a}'_1 - \mathsf{b}'_2\mathsf{a}'_2)\cos2\theta - (\mathsf{b}'_1\mathsf{a}'_2 + \mathsf{b}'_2\mathsf{a}'_1)\sin2\theta\\ &= \boldsymbol{\Sigma}'_1\cos2\theta - \boldsymbol{\Sigma}'_2\sin2\theta\end{aligned} \tag{296.1}$$

$$\begin{aligned}\boldsymbol{\Sigma}_2 = &\quad(\mathsf{b}'_1\cos\theta - \mathsf{b}'_2\sin\theta)(\mathsf{a}'_1\sin\theta + \mathsf{a}'_2\cos\theta)\\ &+(\mathsf{b}'_1\sin\theta + \mathsf{b}'_2\cos\theta)(\mathsf{a}'_1\cos\theta - \mathsf{a}'_2\sin\theta)\\ &= (\mathsf{b}'_1\mathsf{a}'_1 - \mathsf{b}'_2\mathsf{a}'_2)\sin2\theta + (\mathsf{b}'_1\mathsf{a}'_2 + \mathsf{b}'_2\mathsf{a}'_1)\cos2\theta\\ &= \boldsymbol{\Sigma}'_1\sin2\theta + \boldsymbol{\Sigma}'_2\cos2\theta\end{aligned} \tag{296.2}$$

$$\begin{aligned}\boldsymbol{\Sigma}_3 = &\quad i(\mathsf{b}'_1\cos\theta - \mathsf{b}'_2\sin\theta)(\mathsf{a}'_1\sin\theta + \mathsf{a}'_2\cos\theta)\\ &- i(\mathsf{b}'_1\sin\theta + \mathsf{b}'_2\cos\theta)(\mathsf{a}'_1\cos\theta - \mathsf{a}'_2\sin\theta)\\ &= i(\mathsf{b}'_1\mathsf{a}'_2 - \mathsf{b}'_2\mathsf{a}'_1)\\ &= \boldsymbol{\Sigma}'_3\end{aligned} \tag{296.3}$$

From (296) we learn that the quantum mechanical Stokes parameters $(\psi|\boldsymbol{\Sigma}_\mu|\psi)$ respond—for every $|\psi)$—to frame-rotation in precise imitation of the equations (292) to which we were led geometrically/classically. This remark pertains even to states with which no precisely-drawn ellipse can be associated,[102] in which connection we notice that

$$\text{polarization } P = \frac{\sqrt{\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2}}{\Sigma_0} \quad \text{is a rotational invariant}$$

An overt geometrical symmetry of the isotropic oscillator—the rotational symmetry, commonly expressed $[\mathsf{H}, \mathsf{L}_3] = \mathbf{0}$—has found expression in (296) as a

[102] Included in this broad category are the eigenstates $|n_1, n_2)$, which were seen at (289/290) to be only partially polarized unless $n_1 n_2 = 0$.

symmetry (with respect to rotation about the S_3-axis) of the Poincaré sphere. But the symmetry group of the sphere is $O(3)$, not $O(2)$. The "extra symmetry" can be associated with "hidden symmetry" of the dynamical system, by the following line of argument:

The n^{th} energy eigenspace $\mathcal{H}_n$ of the 2-dimensional isotropic oscillator is, as has been remarked, $(n+1)$-dimensional—big enough to support faithful representations of $U(1), U(2), \ldots, U(n+1)$. To say the same thing another way: if $\big\{|n-j,j) : j = 0,1,2,\ldots,n\big\}$ is an orthonormal basis in $\mathcal{H}_n$ as previously described, and if $\big\{|i) : i = 0,1,2,\ldots,n\big\}$ is any other orthonormal basis, then the $(n+1)\times(n+1)$ matrix $\mathbb{U}$ with elements $U^i{}_j \equiv (i|n-j,j)$ is necessarily unitary. A total of $(n+1)^2$ adjustable parameters enter into the specification of $\mathbb{U}$, but only one (θ) into the design of the $\mathbb{R}(\theta;n)$ to which $\mathbb{U}$ reduces by stark specialization.

Familiarly, the general element of $U(2)$ can be described

$$\mathbb{U} = e^{i\varphi}\cdot\mathbb{S}$$

$$\mathbb{S} \equiv \mathbb{S}(\alpha,\beta;1) \equiv \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad : \quad \text{general element of } SU(2)$$

where φ is real and α,β are complex numbers subject to the unimodularity constraint $\alpha^*\alpha + \beta^*\beta = 1$. Suppose, in this light, we were in place of (292), to write

$$\left.\begin{aligned} \mathsf{b}_1 &= \alpha\ \mathsf{b}_1' + \beta\ \mathsf{b}_2' \\ \mathsf{b}_2 &= -\beta^*\mathsf{b}_1' + \alpha^*\mathsf{b}_2' \\ &\Downarrow \\ \mathsf{a}_1 &= \alpha^*\mathsf{a}_1' + \beta^*\mathsf{a}_2' \\ \mathsf{a}_2 &= -\beta\ \mathsf{a}_1' + \alpha\ \mathsf{a}_2' \end{aligned}\right\} \tag{297}$$

We would then be in position to describe *any* adjustment of the form

$$\text{orthonormal basis} \longrightarrow \text{orthonormal basis}$$

within $\mathcal{H}_1$, and by equations of the form[103]

$$\begin{pmatrix} \frac{1}{\sqrt{2!0!}}\mathsf{b}_1^2\mathsf{b}_2^0 \\ \frac{1}{\sqrt{1!1!}}\mathsf{b}_1^1\mathsf{b}_2^1 \\ \frac{1}{\sqrt{0!2!}}\mathsf{b}_1^0\mathsf{b}_2^2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\beta^* & (\alpha^*\alpha - \beta^*\beta) & \sqrt{2}\beta\alpha^* \\ \beta^{*2} & -\sqrt{2}\alpha^*\beta^* & \alpha^{*2} \end{pmatrix}}_{\mathbb{S}(\alpha,\beta;2)} \begin{pmatrix} \frac{1}{\sqrt{2!0!}}{\mathsf{b}_1'}^2{\mathsf{b}_2'}^0 \\ \frac{1}{\sqrt{1!1!}}{\mathsf{b}_1'}^1{\mathsf{b}_2'}^1 \\ \frac{1}{\sqrt{0!2!}}{\mathsf{b}_1'}^0{\mathsf{b}_2'}^2 \end{pmatrix}$$

would set up "echos" of that adjustment within $\mathcal{H}_2$,$\mathcal{H}_3$,... By computation (consigned to *Mathematica*) we discover that $\mathbb{S}(\alpha,\beta;2)$ is unimodular

$$\det\mathbb{S}(\alpha,\beta;2) = (\alpha^*\alpha + \beta^*\beta)^3 = 1$$

[103] The following equation is an enlargement upon the essence of (294.2), which can be recovered as a special case, and by generalization one is led to a similar enlargement upon (295).

and unitary

$$\mathbb{S}^\dagger(\alpha,\beta;2)\,\mathbb{S}(\alpha,\beta;2)=\mathbb{I}$$

and, moreover, that such matrices compose by a rule

$$\mathbb{S}(\alpha_1,\beta_1;2)\,\mathbb{S}(\alpha_2,\beta_2;2)=\mathbb{S}(\alpha_1\alpha_2-\beta_1\beta_2^*,\alpha_1\beta_2+\beta_1\alpha_2^*;2)$$

which precisely mimics that of $\mathbb{S}(\alpha,\beta;1)$:

$$\mathbb{S}(\alpha_1,\beta_1;1)\,\mathbb{S}(\alpha_2,\beta_2;1)=\mathbb{S}(\alpha_1\alpha_2-\beta_1\beta_2^*,\alpha_1\beta_2+\beta_1\alpha_2^*;1) \tag{298}$$

We anticipate on this basis that the matrices $\mathbb{S}(\alpha,\beta;n)$ could be shown to provide an $(n+1)$-dimensional unimodular unitary representation of $SU(2)$. Finally, we introduce (297) into (262) and obtain an enlargement upon (296)

$$\begin{aligned}
\mathbf{\Sigma}_0 &= \quad (\bar{\alpha}\alpha+\bar{\beta}\beta)\mathbf{\Sigma}_0'\\
\mathbf{\Sigma}_1 &= +\,(\bar{\alpha}\alpha-\bar{\beta}\beta)\mathbf{\Sigma}_1'+\ (\alpha\bar{\beta}+\beta\bar{\alpha})\mathbf{\Sigma}_2'-i(\alpha\bar{\beta}-\beta\bar{\alpha})\mathbf{\Sigma}_3'\\
\mathbf{\Sigma}_2 &= -\,(\alpha\beta+\bar{\alpha}\bar{\beta})\mathbf{\Sigma}_1'+\ \tfrac{1}{2}(\alpha^2+\bar{\alpha}^2-\beta^2-\bar{\beta}^2)\mathbf{\Sigma}_2'-i\tfrac{1}{2}(\alpha^2-\bar{\alpha}^2+\beta^2-\bar{\beta}^2)\mathbf{\Sigma}_3'\\
\mathbf{\Sigma}_3 &= -i(\alpha\beta-\bar{\alpha}\bar{\beta})\mathbf{\Sigma}_1'+i\tfrac{1}{2}(\alpha^2-\bar{\alpha}^2-\beta^2+\bar{\beta}^2)\mathbf{\Sigma}_2'+\ \tfrac{1}{2}(\alpha^2+\bar{\alpha}^2+\beta^2+\bar{\beta}^2)\mathbf{\Sigma}_3'
\end{aligned}$$

which can be abbreviated

$$\begin{pmatrix}\mathbf{\Sigma}_0\\ \mathbf{\Sigma}_1\\ \mathbf{\Sigma}_2\\ \mathbf{\Sigma}_3\end{pmatrix}=\begin{pmatrix}1&0&0&0\\ 0&R^1{}_1&R^1{}_1&R^1{}_1\\ 0&R^2{}_1&R^2{}_2&R^2{}_3\\ 0&R^3{}_1&R^3{}_2&R^3{}_3\end{pmatrix}\begin{pmatrix}\mathbf{\Sigma}_0'\\ \mathbf{\Sigma}_1'\\ \mathbf{\Sigma}_2'\\ \mathbf{\Sigma}_3'\end{pmatrix} \tag{299}$$

Computation shows the real 3×3 matrix $\mathbb{R}(\alpha,\beta)\equiv\|R^i{}_j(\alpha,\beta)\|$ to be a proper rotation matrix, and to compose by the rule

$$\mathbb{R}(\alpha_1,\beta_1)\,\mathbb{R}(\alpha_2,\beta_2)=\mathbb{R}(\alpha_1\alpha_2-\beta_1\beta_2^*,\alpha_1\beta_2+\beta_1\alpha_2^*)$$

A little experimentation, inspired by remarks which assume importance in §3 of "Algebraic theory of spherical harmonics" (1996), leads finally to the curious observation that $\mathbb{R}(\alpha,\beta)$ can be described

$$\mathbb{R}(\alpha,\beta)=\mathbb{C}^\dagger\,\mathbb{S}(\alpha,\beta;2)\,\mathbb{C}\quad\text{with}\quad \mathbb{C}\equiv\tfrac{1}{\sqrt{2}}\begin{pmatrix}0&i&1\\ -\sqrt{2}\,i&0&0\\ 0&-i&1\end{pmatrix}$$

We have been led back once again to the familiar association

$$SU(2)\longleftarrow\!\!\longrightarrow O(3)\quad\text{ i.e. }\quad \pm\,\mathbb{S}(\alpha,\beta;1)\longleftrightarrow\mathbb{R}(\alpha,\beta)$$

but now in a somewhat unfamiliar setting: we have found that when the respective components $|\psi_n)$ of the isotropic oscillator state

$$|\psi)=\sum_n|\psi_n)\quad\text{with}\quad|\psi_n)\in\mathcal{H}_n$$

are adjusted in concerted response to (vi) (which is to say: in representation of $SU(2)$) then the Stokes parameters $(\psi|\boldsymbol{\Sigma}_\mu|\psi)$ experience a rotation—namely, the rotation of which $\{\alpha, \beta\}$ are the so-called "Cayley-Klein parameters." More interesting is the fact that although

$$\text{angular momentum of isotropic oscillator} = \text{integer}\cdot\hbar$$

the theory as a whole provides an embodiment of the quantum theory of spin (spinor representations of the rotation group). It becomes plausible in this light to anticipate that ladder operators borrowed from oscillator theory can be made to support a systematic account of the quantum theory of angular momentum, and no surprise to discover that Julian Schwinger once proceeded down precisely that road;[104] a "hidden symmetry" has in this instance become the symmetry of principal interest, and has been pressed into fruitful service.

21. Quantum mechanical Foucault pendulum and geometric phase. Formally (and superficially), light beams and isotropic oscillators appear to have much in common. But to position oneself at $\boldsymbol{x}$ interpret the wavefunction $\psi(\boldsymbol{x}, t)$ —whether of an oscillator or of *any* quantum system—as a "signal" one would have to suspend the quantum theory of measurement. One might be tempted to associate an *ensemble* of such wave functions with a "stream of photons," but the photon is a slippery beast, found to be the slipperier the more closely it is examined.[105] Nevertheless. . . the classical theory of monochromatic light beams gave us Pancharatnam's phase (§8), and it seems plausible that the classical/quantum mechanics of isotropic oscillators might support an analogous notion. That is the question I propose now to explore.

Of course, mention of "geometrical phase" in connection with a quantum mechanical problem brings first to mind "Berry's phase." David Griffiths has given a useful introductory account of that subject,[106] but introduces that discussion with an account of the "Foucault pendulum problem" which derives some of its substantial interest from the circumstance that it has, in fact, *little to do either with Berry's phase or with its classical analog ("Hannay's phase").* The latter (see the following figure) refer to the *temporal* phase shifts which typically result from adiabatic transport around loops drawn in *parameter space* (the space of parameters which enter into the design of the Hamiltonian), while the former refers to the *adjusted orbital orientation*—the "orbital phase" I will call it, to achieve a kind of terminological symmetry—which typically results from adiabatic transport around loops drawn in *physical space*. Temporal phase shifts may, indeed, result from "Foucault processes," but (in the presence of orbital adjustment become somewhat problematic to define, and in any event)

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[104] "On angular momentum," U. S. Atomic Energy Commission Publication NYO-3071 (1952).

[105] See Chapters 10–12 in L. Mandel & E. Wolf, *Optical Coherence & Quantum Optics* (1995).

[106] *Introduction to Quantum Mechanics* (1994), §10.2.

comprise an aspect of the "Foucault problem" to which Foucault[107] himself paid no observational attention.

A word about Hannay's accomplishment before we get on with our main work: Hannay's objective[108] was to identify a direct classical analog of Berry's phase. In imitation of Berry's assumption (sufficient in itself to exclude isotropic oscillators, except in their ground states) that the quantum system under discussion is in a non-degenerate eigenstate, Hannay looks to classical systems with only a single cyclic degree of freedom (which—if for a different reason—again excludes isotropic oscillators). With Berry, he contemplates loop-like adiabatic "excursions in the space of control parameters," and looks for the consequent adjustment in the *temporal* phase of the system (see the following figure). The possibility of an "orbital phase" Hannay was careful to exclude by initial presumption, the better to expose the effect of interest to him. On the other hand, "orbital phase" was precisely and exclusively the effect of interest to Foucault; "temporal phase" is a detail which Foucault was tacitly content to set aside (as also, in another connection but at about the same time, was Stokes), and concerning which Griffiths has, in fact, nothing to say.[109] Griffiths' allusion to "Hannay phase" is misguided; the value of his remarks lies elsewhere.

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[107] Jean Bernard Léon Foucault (1819–1868)—son of a bookseller, small and frail—had been an indifferent student, but developed into an experimentalist of exceptional skill, and an influential popular expositor of science. In many respects his career reminds one of Michael Faraday. During the 1840's he, with his frequent collaborator Fizeau, did pioneering work in astrophotography. In the early 1850's they were both concerned with benchtop measurements of the speed of light (in air, water). The "Foucault pendulum" was a by-product of an attempt (1851) to place a conical pendulum at the heart of a telescope drive, and was followed a year later by the invention of the gyroscope; both permitted benchtop study of the earth's rotation, and stimulated the further development of theoretical mechanics. (The theory of the Foucault pendulum had been worked out already in 1838 by Poisson, who, however, considered the effect to be unobservably small.) Foucault also invented techniques for producing and testing telescope mirrors which have become classic.

[108] J. H. Hannay, "Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian," J. Phys. A: Math. Gen. **18**, 221 (1985), reprinted at p. 426 in Shapere & Wilczek. Hannay was a colleague of Berry's at the University of Bristol. His work is couched in the language of Hamilton-Jacobi theory (action and angle variables), which has been recognized since early in the century (Lorentz, Einstein, Ehrenfest & Bergers; see Max Jammers, *The Conceptual Development of Quantum Mechanics* (1966), §3.1) to be the language of choice for discussion of classical adiabatic approximation theory. Hannay's work was elaborated (in advance of publication) by Berry himself: "Classical adiabatic angles and quantal adiabatic phase," J. Phys. A: Math. Gen. **18**, 15 (1985), reprinted at p.436 in Shapere & Wilczek.

[109] *Relative* temporal phase lies, on the other hand, at the heart of the issue addressed by Pancharatnam.

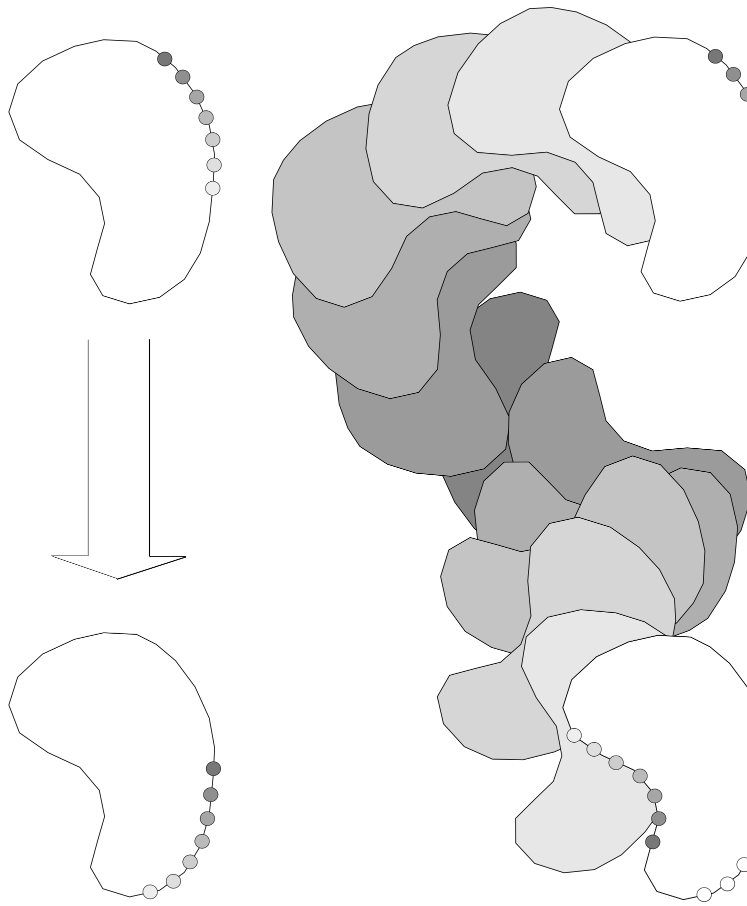


FIGURE 22: *Hannay's problem. In upper left, a bead (more usefully: a population of beads) slides on a wire loop. At upper right, a copy of the system is adiabatically deformed, but returns ultimately to its original design. The problem: what is the phase of the population at lower right relative to that of the population (lower left) which has experienced no such adventure?*

My objective in the following remarks is to proceed stepwise to a deepened statement of "Foucault's problem," and to indication of how Griffiths proposes to "geometrize" the problem. Let the earth's rotation be, for the moment, suspended. Picture Jean Foucault, as he

- makes a daguerreotype record of the orbit being traced by his pendulum;
- hikes—pendulum in hand—slowly and with exquisite care about the French countryside;
- returns to his Parisian laboratory, where he compares the orbit-at-return with the recorded orbit-at-departure.

(Griffiths, for his own mathematical convenience, would have Foucault hike straight south from the North Pole to the equator, then along the equator to a point of different longitude, then return straight north to the pole, where he would compare the orbit of the pendulum in hand to the orbit which he had traced on the ice, prior to his departure. Foucault himself chose simply to sit in his laboratory, and let the rotating earth transport him along a circle of constant latitude; by this strategy he was able, with much reduced effort, to *monitor* the precession of his pendulum; i.e., to observe its temporal progress from initial to final orientation.) Foucault's conical pendulum is, by careful design, an isotropic oscillator, activated by gravity. If activated internally, by an isotropic spring, then gravity could be "turned off;" in hiking about the French countryside Foucault, if he sought to duplicate his former experiment, would then have to exercise care to keep the oscillatory plane always tangent to the (former) geosphere. We, a century and a half later, might contemplate doing a similar thing here in the laboratory with a *quantum* oscillator, tangently transported around a loop inscribed (not necessarily on a sphere, but) on an arbitrary surface (fender of an automobile).[110]

Griffiths, on the intuitively compelling evidence of a class of special cases (Figure 23), considers it to be self-evident that

$$\text{adiabatic transport} \Longrightarrow \text{parallel transport} \tag{300}$$

and in support of that proposition advances what he calls a "purely geometrical interpretation" of the familiar Foucault precession formula—a formula which, as he correctly remarks, is "ordinarily obtained by appeal to Coriolis forces in the rotating reference frame."[111] But whatever may be the intuitive appeal of (300), and its effectiveness in special cases, it stands at the moment as a bald assertion. We have acquired an obligation to inquire into the physical credentials of (300). This Griffiths—in hot pursuit of Berry phase—does not linger to do; this I now undertake to do. On the right we encounter a mathematical notion which is secure but non-trivial,[112] and on the left a physical notion which, to a surprising degree, remains elusive; Griffiths (in his §10.1.2) works within a context so sharply defined as to permit him to speak of an "adiabatic theorem," but

[110] At a still higher level of abstraction—abandoning on the one hand the distinguishing physical features of isotropic oscillators and, on the other, the distinguishing geometrical features of loops inscribed on surfaces—we might imagine ourselves to be in possession of two identical physical systems in initally identical states, might take one of those systems on an adiabatic excursion, and ask: How do the states of the two systems compare when they are brought back together?

[111] See, for example, J. B. Marion, *Classical Dynamics of Particles & Systems* (2nd edition 1970), p. 355; or T. W. B. Kibble, *Classical Mechanics* (2nd edition 1973), §5.4.

[112] I refer to the *Gauss-Bonnet theorem*, concerning which Gauss (1827) wrote that "this theorem, if I mistake not, ought to be counted among the most elegant in the theory of curved surfaces."

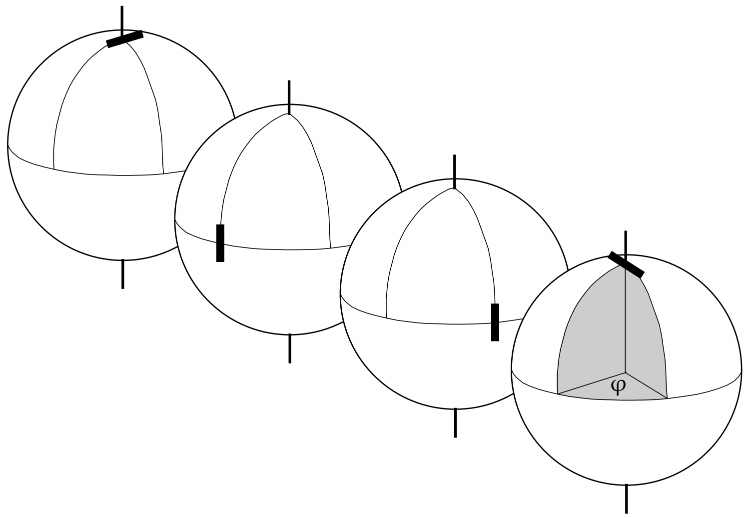


FIGURE 23: *"Griffiths' tour:" a pendulum is transported very gently (which is to say: adiabatically)*
- *from North Pole to equator;*
- *from equatorial point to an equatorial point of different longitude (let φ denote the longitudinal difference);*
- *back to the North Pole*

of a non-rotating earth.

Hannay, citing the experience of V. I. Arnold,[113] was careful to emphasize that while "...the adiabatic principle is well defined and widely realized physically, it appears to be surprisingly difficult to eliminate the mathematical loopholes which prevent the simple statement that it holds rigorously in the limit of slow change." Einstein, in recognition of this circumstance, prefered to speak of the "adiabatic hypothesis," the validity of which Hannay is prepared to "take for granted." It is in an effort to clarify aspects of the situation that I look now to the dynamical details of a couple of simple systems.

Look first to the system described in Figure 24. From

$$\left.\begin{aligned} X &= R\cos\theta - y\sin\theta \\ Y &= R\sin\theta + y\cos\theta \end{aligned}\right\} \tag{301}$$

we obtain

$$\left.\begin{aligned} \dot{X} &= -R\dot{\theta}\sin\theta - y\dot{\theta}\cos\theta - \dot{y}\sin\theta \\ \dot{Y} &= +R\dot{\theta}\cos\theta - y\dot{\theta}\sin\theta + \dot{y}\cos\theta \end{aligned}\right\} \tag{302}$$

[113] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (2nd edition 1989), Chapter 10. It was, by the way, from Arnold that Hannay borrowed his figure.

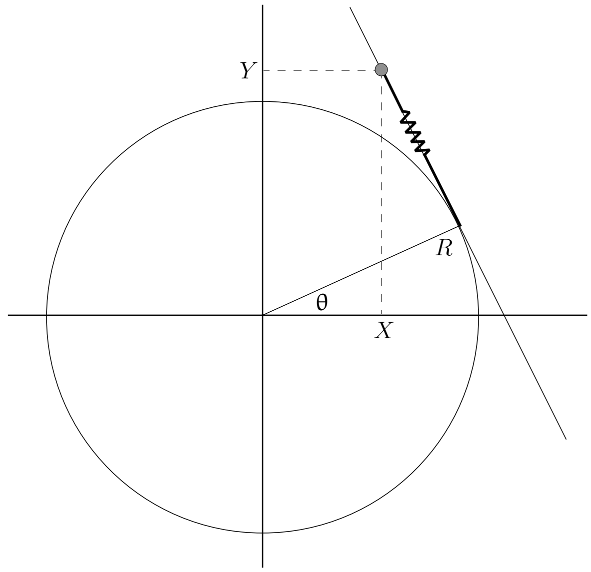


FIGURE 24: *A particle of mass m is constrained to move on a line tangent to a circle. It is attached by a spring to the point of tangency, and that point moves under external control $\theta(t)$. The system has a single degree of freedom, which is taken to be*

$$y \equiv \textit{distance from point to tangency to the mass point}$$

The radius R is held constant, though it could be promoted to the status of a control parameter.

giving

$$\dot{X}^2 + \dot{Y}^2 = \dot{y}^2 + 2R\dot{y}\dot{\theta} + (R^2 + y^2)\dot{\theta}^2 \tag{303}$$

So the Lagrangian becomes

$$L = \tfrac{1}{2}m(\dot{y}^2 - \omega^2 y^2) + \underbrace{\tfrac{1}{2}m\dot{\theta}^2 y^2}_{\text{centrifugal}} + \underbrace{mR\dot{\theta}\,\dot{y}}_{\text{Coriolis}} + \underbrace{\tfrac{1}{2}mR^2\dot{\theta}^2}_{\text{gauge}} \tag{304}$$

and the equation of motion reads

$$\ddot{y} + \omega^2(t)y = -R\ddot{\theta}(t) \quad \text{with} \quad \omega^2(t) \equiv \omega^2 - \dot{\theta}^2(t) \tag{305}$$

Under conditions which permit the expression on the right to be neglected, this becomes the equation of a parametrically stimulated oscillator, concerning which a great deal has been written.[114] The effect of $\dot{\theta}^2 < \omega^2$ is to "soften

[114] For an excellent review, see Chapter 5 of A. H. Nayfeh & D. T. Mook, *Nonlinear Oscillations* (1979)

the spring" (prolong the period). If $\dot{\theta}^2 = \omega^2$ holds (not just momentarily but) constantly in time, then the motion of the particle is in effect (i.e., with respect to its non-inertial frame) "free," while if $\dot{\theta}^2 > \omega^2$ the "effective spring" has become repulsive.[115] It seems clear, even in the absence of detailed discussion, that (because $\dot{\theta}(t)$ enters squared into the construction of $\omega^2(t)$) a transported oscillator, thought of as a "clock," always runs slow with respect to a stationary oscillator, but that the effect evaporates in the adiabatic limit. But if we promote R to the status of a control parameter then we have

$$\dot{X}^2 + \dot{Y}^2 = \dot{y}^2 + 2R\dot{y}\dot{\theta} + (R^2 + y^2)\dot{\theta}^2 + \dot{R}^2 - 2\dot{R}\dot{\theta}\,y$$

giving

$$L = \tfrac{1}{2}m(\dot{y}^2 - \omega^2 y^2) + \underbrace{\tfrac{1}{2}m\big(\dot{\theta}^2 y^2 - 2\dot{R}\dot{\theta}y\big)}_{\text{centrifugal}} + \underbrace{mR\dot{\theta}\dot{y}}_{\text{Coriolis}} + \underbrace{\tfrac{1}{2}m\big(R^2\dot{\theta}^2 + \dot{R}^2\big)}_{\text{gauge}}$$

whence $\ddot{y} + \omega^2(t)y = -R(t)\ddot{\theta}(t)$ and it becomes possible to contemplate *loops* in parameter space; according to Hannay an effect then *may* survive, even in the adiabatic limit. Such an effect, if present, would, as I have several times emphasized, possess the character of a "temporal phase;" the system has too few dimensions to support the notion of "orbital phase."

While the system just considered—the "Foucault pendulum in flatland"—is of some independent interest, is has been presented as methodological warm-up for the discussion to which I now turn. The physical problem now before us is described in Figure 25. To obtain the $\{x, y, z\}$-frame from the $\{X, Y, Z\}$-frame one can proceed stepwise, as follows:

- execute a righthanded ϕ-rotation about the Z-axis;
- execute a lefthanded θ-rotation about the repositioned Y-axis;
- translate a distance R along the newly repositioned X-axis;
- adjust coordinate names: $X''' \to z$, $Y''' \to x$, $Z''' \to y$.

Thus—by a procedure reminiscent of that used to define Euler's angles—do we obtain

$$\begin{aligned}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z+R \end{pmatrix} \\ &= \begin{pmatrix} (R+z)\cos\theta\cos\phi - y\sin\theta\cos\phi - x\sin\phi \\ (R+z)\cos\theta\sin\phi - y\sin\theta\sin\phi + x\cos\phi \\ (R+z)\sin\theta + y\cos\theta \end{pmatrix}\end{aligned}$$

[115] For more detailed discussion of the physics of the situation, which proceeds from

$$H = \tfrac{1}{2m}\big[p - A\big]^2 + \tfrac{1}{2}m(\omega^2 - \dot{\theta}^2)y^2$$
$$A \equiv mR\dot{\theta} \ : \ \text{very like an electromagnetic potential}$$

see CLASSICAL MECHANICS (1983), pp. 413–418. Note that the existence of a Hamiltonian creates the possibility of quantum mechanical treatment.

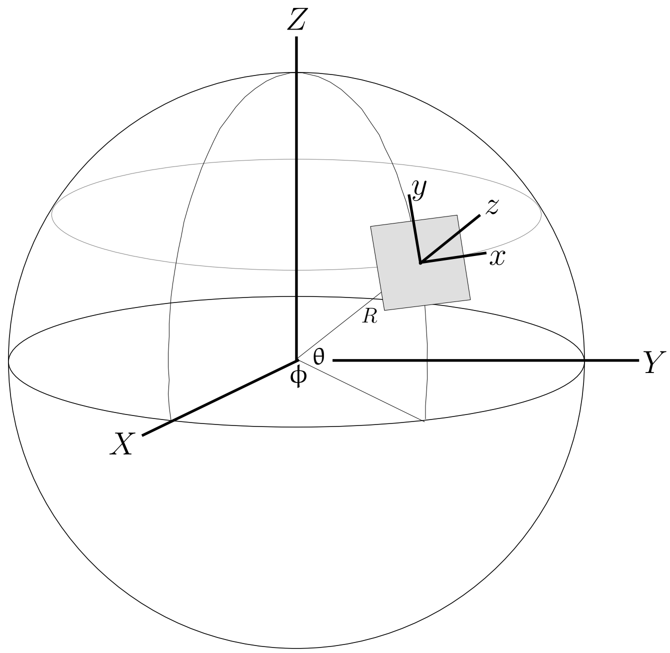


FIGURE 25: *Go to non-polar point* $\{\phi,\theta\}$ *on a sphere of radius* R. *Erect a Cartesian frame, so oriented that the* x*-axis points east, the* y*-axis points north, the* z*-axis points up. The plane* $z=0$ *is locally tangent to the sphere, and* $x=y=0$ *marks its point of tangency. Let a mass* m *be attached to the point of tangency by an isotropic spring. Launch* ϕ *and* θ *into slow motion, and require* m *to move subject to the constraint* $z(t)=0$. *We have interest in the adiabatic properties of the "generalized Foucault pendulum" thus defined.*

which, it will be observed, positions the $\{x,y,z\}$-origin at $\begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ R\sin\theta \end{pmatrix}$—as expected/required—and gives back (301) at $x=z=\phi=0$. Setting $z=0$ we find that points $\{x,y\}$ on the tangent plane lie at points

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} R\cos\theta\cos\phi - y\sin\theta\cos\phi - x\sin\phi \\ R\cos\theta\sin\phi - y\sin\theta\sin\phi + x\cos\phi \\ R\sin\theta + y\cos\theta \end{pmatrix} \tag{306}$$

in the enveloping 3-space.

To tickle out the significance of (306) we look to a graded series of special cases, and entrust the computational details—which can become exceedingly laborious—to *Mathematica*. If, in the simplest case, ϕ and θ are held constant then (306) gives

$$\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 = \dot{x}^2 + \dot{y}^2$$

whence $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}m\omega^2(x^2+y^2)$, with all the familiar consequences. To develop physics associated with first and third legs of "Griffiths' tour" (Figure 23) we assign time-dependence to θ and an arbitrary constant value to the longitude ϕ; then

$$\dot{X}^2+\dot{Y}^2+\dot{Z}^2 = \dot{x}^2+\dot{y}^2+2R\dot{\theta}\,\dot{y}+(R^2+y^2)\dot{\theta}^2 \tag{307.1}$$

gives

$$L = \tfrac{1}{2}m\big(\dot{x}^2-\omega^2x^2\big)+\tfrac{1}{2}m\big(\dot{y}^2-[\omega^2-\dot{\theta}^2]y^2+2R\dot{\theta}\,\dot{y}\big)+\tfrac{1}{2}mR^2\dot{\theta}^2$$

The resulting equations of motion

$$\left.\begin{aligned}\ddot{x}+\omega^2x&=0\\ \ddot{y}+[\omega^2-\dot{\theta}^2]y&=-R\ddot{\theta}\end{aligned}\right\} \tag{307.2}$$

are again *uncoupled*, and the latter equation reproduces (305). The second (equatorial) leg of Griffiths' tour leads us, on the other hand, to set $\theta=0$ and to assign the time-dependence to ϕ; then

$$\dot{X}^2+\dot{Y}^2+\dot{Z}^2 = \dot{x}^2+\dot{y}^2+2R\dot{\phi}\,\dot{x}+(R^2+x^2)\dot{\phi}^2 \tag{308.1}$$

gives

$$L = \tfrac{1}{2}m\big(\dot{y}^2-\omega^2y^2\big)+\tfrac{1}{2}m\big(\dot{x}^2-[\omega^2-\dot{\phi}^2]x^2+2R\dot{\phi}\,\dot{x}\big)+\tfrac{1}{2}mR^2\dot{\phi}^2$$

whence

$$\left.\begin{aligned}\ddot{x}+[\omega^2-\dot{\phi}^2]x&=-R\ddot{\phi}\\ \ddot{y}+\omega^2y&=0\end{aligned}\right\} \tag{308.2}$$

All three legs are geodesic, and all three have been seen to give rise to the same physics; namely, the physics to which we were led in discussion pursuant to Figure 24.

Preceeding remarks serve to substantiate Griffiths' intuition so far as it relates to tours of the specialized type illustrated in Figure 23, but their true significance lies deeper: since—by proper pre-positioning of the underlying spherical coordinate system (which, on a non-rotating earth, is certainly our option)—any geodesic can be taken to be "an equator" (alternatively: a great circle of constant longitude), we have in effect established that *transport along any great circle leads to uncoupled equations of oscillator motion—equations which in the adiabatic limit retain no reference to the non-inertiality of the local Cartesian frame*. This result serves to establish (300) for *geodesic* transport, and therefore for adiabatic transport along any spline curve with geodesic segments…which is (in the limit) to say: for *any* nice curve inscribed on the sphere.

It would, however, be inelegant to realize Foucault's "circle of constant latitude" as the limit of a sequence of "geodesic polygons" (order $p\uparrow\infty$); there are more efficient—and more instructive—ways to proceed: if we assign time-dependence to ϕ, and a constant but now *non-zero* value to θ, then by computation

$$\begin{aligned}\dot{X}^2+\dot{Y}^2+\dot{Z}^2=\dot{x}^2+\dot{y}^2&+2R\dot{\phi}\,\dot{x}\cos\theta\\&+2(x\dot{y}-y\dot{x})\dot{\phi}\sin\theta\\&+\dot{\phi}^2\big(x^2+\tfrac{1}{2}[1-\cos 2\theta]y^2\big)\\&-yR\dot{\phi}^2\sin 2\theta\\&+\tfrac{1}{2}R^2\dot{\phi}^2[1+\cos 2\theta]\end{aligned}\tag{309}$$

which gives back (308.1) in the case $\theta=0$ and leads to *coupled* equations of motion

$$\begin{aligned}\ddot{x}+\tfrac{d}{dt}\big[-y\dot{\phi}\sin\theta+R\dot{\phi}\cos\theta\big]+\big(\omega^2-\dot{\phi}^2\big)x-\dot{y}\dot{\phi}\sin\theta=0\\ \ddot{y}+\tfrac{d}{dt}\big[+x\dot{\phi}\sin\theta\big]+\big(\omega^2-\tfrac{1}{2}[1-\cos 2\theta]\dot{\phi}^2\big)y+\dot{x}\dot{\phi}\sin\theta=-R\dot{\phi}^2\sin 2\theta\end{aligned}$$

which at $\theta=0$ give back (308.2). The case of physical interest to Foucault arises when $\dot{\phi}$ is constant; we then have

$$\left.\begin{aligned}\ddot{x}-2\dot{y}\,\dot{\phi}\sin\theta+\big(\omega^2-\dot{\phi}^2\big)x=0\\ \ddot{y}+2\dot{x}\,\dot{\phi}\sin\theta+\big(\omega^2-\tfrac{1}{2}[1-\cos 2\theta]\dot{\phi}^2\big)y=-R\dot{\phi}^2\sin 2\theta\end{aligned}\right\}\tag{310}$$

which, in the approximation that $\dot{\phi}$ is small, become

$$\left.\begin{aligned}\ddot{x}-2\Omega\dot{y}+\omega^2x\approx 0\\ \ddot{y}+2\Omega\dot{x}+\omega^2y\approx 0\end{aligned}\right\}\tag{311}$$

where Ω is the latitude-dependent constant defined $\Omega(\theta)\equiv\dot{\phi}\sin\theta$. These are precisely the equations which in most elementary texts[111] stand central to the analysis of Foucault's problem; the standard procedure is to write

$$\ddot{z}+2i\Omega\dot{z}+\omega^2z=0$$

and to look for solutions of the form $z(t)=Ae^{pt}$. From $p^2+2i\Omega p+\omega^2=0$ one is led to $p=-i\Omega\pm i\tilde{\omega}$ with $\tilde{\omega}\equiv\sqrt{\omega^2+\Omega^2}\approx\omega$, and to a general solution which can (in that same adiabatic approximation) be deployed

$$z(t)=\mathcal{X}e^{-i\Omega t}\cos\omega t+i\,\mathcal{Y}e^{-i\Omega t}\cos(\omega t+\delta)$$

giving[116]

$$\left.\begin{aligned}x(t)&=\quad\mathcal{X}\cos\Omega t\cos\omega t+\mathcal{Y}\,\sin\Omega t\cos(\omega t+\delta)\\ y(t)&=-\mathcal{X}\,\sin\Omega t\cos\omega t+\mathcal{Y}\,\cos\Omega t\cos(\omega t+\delta)\end{aligned}\right\}\tag{312}$$

---

[116] The following equation is (for reasons having mainly to do with some experimental conventions) usually encountered only in the "linearly polarized" form obtained (say) by setting $\mathcal{Y}=0$, which Foucault accomplished by invention of the clever "burned string trick."

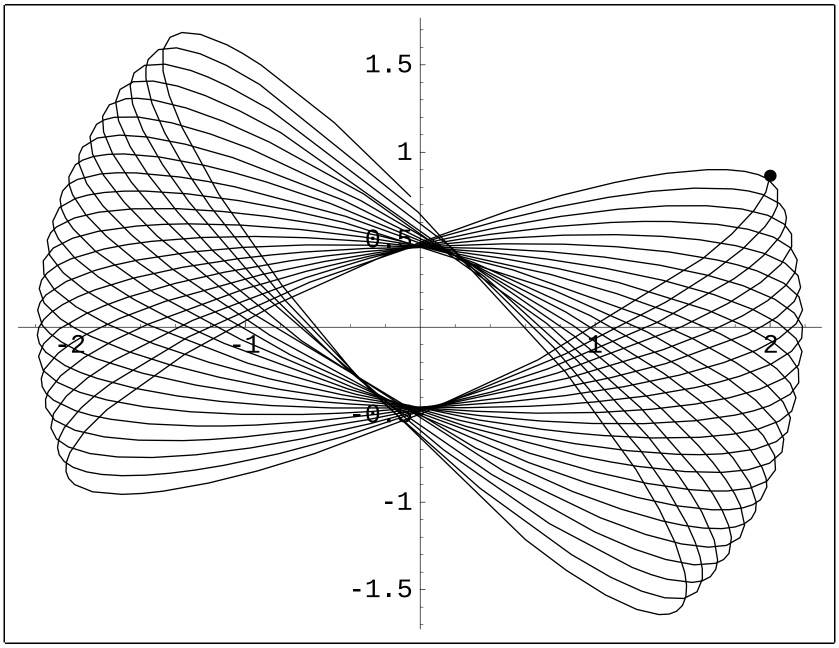


FIGURE 26: *Curve traced by (314) in the case* $\mathfrak{X} = 2$, $\mathfrak{Y} = 1$, $\delta = \frac{\pi}{6}$, $\Omega = \frac{1}{100}\omega$. *A dot marks the launch point. Note the sense of the precession.*

Figure 26 shows a curve typical of those to which the preceding equations refer. Equations (311) could have been obtained from this adiabatic approximation

$$L = \tfrac{1}{2}m\Big\{\big(\dot{x}^2 + \dot{y}^2\big) - \omega^2\big(x^2 + y^2\big)\Big\} + \underbrace{m\Omega\big(x\dot{y} - y\dot{x}\big)}_{\text{gyroscopic term}} \tag{313}$$

to the true Lagrangian; note that it is the "gyroscopic term" which is responsible for the coupling (which is to say: for the precession).

The Foucault system described above completes one "trip around the world" in time τ given by $\dot{\phi}\tau = 2\pi$, so we have

$$\text{diurnal precession} = \Omega\tau = 2\pi\sin\theta \tag{314}$$

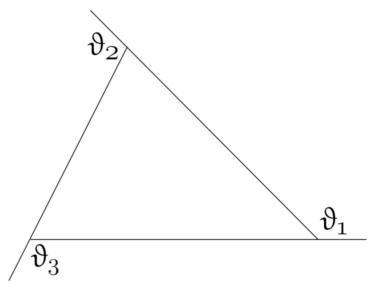
Curiously, this result is indepentent of the "earth's radius" R, and persists even in the limit $R \downarrow 0$.

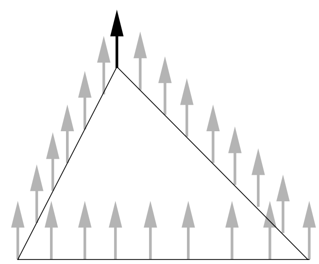
Though we have already in hand the results we will need, I remark, before taking leave of this subject, that we are in position now to assign arbitrary time-dependence simultaneously to *both* ϕ and θ, and thus to transport our oscillator—whether slowly or briskly—along an *arbitrary* spherical curve. We would to led to a still more complicated variant of (309), and to correspondingly more complicated equations of motion, which would simplify in the adiabatic limit but which we could not expect to be able to solve, except in favorable cases or numerically. Geometrical phase theory derives its interest in part from the fact that (as do conservation laws) it permits one nevertheless to say useful general things *about* the solutions.

I turn now from dynamics to geometry. It is elementary that the angles interior to a plane triangle sum to π, therefore that their complements ϑ_i sum to 2π. And (by a simple argument) that

$$\text{sum of exterior angles} = 2\pi$$

since true for triangles, pertains also to simple p-gons ($p \geqslant 2$). And that angular data supplies no information concerning the size of a plane figure. And that parallel transport around a plane triangle (as around any closed curve inscribed on the Euclidean plane) returns an arrow to its original position. These facts,





illustrated in the preceding diagrams, derive their present interest from the circumstance that, in the company of many others, they are *special to the Euclidean plane*. For example, for a *spherical* triangle (bounded, or course, by geodesics, which have become great circles) the sum of the interior angles *exceeds* π, by an amount proportional to the *area* of the triangle;[117] this classic fact

$$\text{area} = R^2\big\{(\text{sum of interior angles}) - \pi\big\}$$

is, for our purposes, most conveniently expressed

$$\begin{aligned} \text{sum of exterior angles} &= 2\pi - \frac{\text{area}}{R^2} \\ &= 2\pi - \Omega \end{aligned} \tag{315}$$

where Ω is the spherical angle subtended at the center of the sphere. The

$$\begin{aligned} \text{“angle excess”} &\equiv \text{sum of interior angles} - \pi \\ &= 2\pi - \text{sum of exterior angles} \\ &= \Omega \end{aligned} \tag{316}$$

[117] Note that a “natural length” can be associated with figures inscribed on a sphere, but none with figures inscribed on the plane.

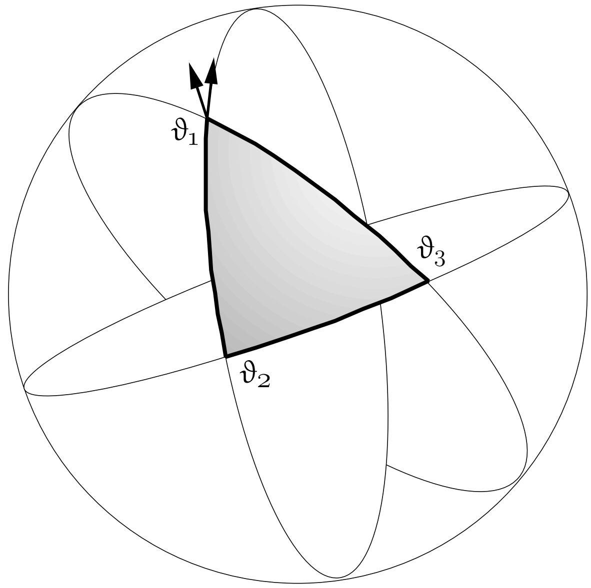


FIGURE 27: *Parallel propagation around a spherical triangle brings about a*

$$\begin{aligned}\text{misalignment} &= 2\pi - (\vartheta_1 + \vartheta_2 + \vartheta_3)\\ &= \Omega \ : \ \text{angular area}\end{aligned}$$

"Griffiths' tour" (Figure 23) entails $\vartheta_2 = \vartheta_3 = \frac{1}{2\pi}$, *and therefore gives a* misalignment $= \pi - \vartheta_1 = \varphi$.

becomes manifest also as the "angular misalignment" associated with parallel propagation around such a curve (see the preceding figure). Equation (316), by straightforward extension, pertains also to "spherical p-gons" ($p \geqslant 2$), which in the case $p = 2$ become "lunes;" the vertices of a lune are necessarily diametric, and the exterior angles necessarily equal: $\vartheta_1 = \vartheta_2 \equiv \vartheta$. Parallel propagation around a lune (which can be thought of as two "Griffiths' tours" joined base to base) brings about a misalignment given by $2(\pi - \vartheta)$.

The Gauss-Bonnet theorem[118] comes into play when one relaxes the requirement that the curves joining the vertices (if any) be geodesic, and permits one to relax also the presumption that the closed curve has been inscribed on a sphere; it is, as it relates to spheres, illustrated in Figure 28, and is for our purposes most usefully expressed

$$\int_C \kappa_g \, ds + \Omega = 2\pi - (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n) \tag{317}$$

[118] For the stark essentials, see the *Encyclopedic Dictionary of Mathematics* (1993), p. 1732. For an accessible account of the details, and many instructive applications, see John McCleary, *Geometry from a Differentiable Viewpoint* (1994), Chapter 12.

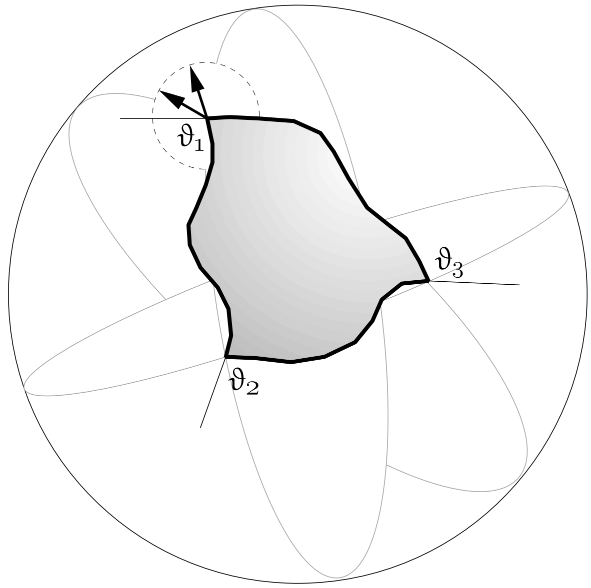


FIGURE 28: *Gauss-Bonnet theorem, as it pertains to spheres. Arcs —which may but need not be geodesic—are joined head to tail (subtending exterior angles ϑ_i) in such a way as to yield a simple closed curve. Let D be the enclosed region, and $C \equiv \partial D$ denote the bounding curve. The theorem asserts that*

$$\int_C \kappa_g\, ds + \iint_D K\, dS = 2\pi - (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)$$

where κ_g is the "geodesic curvature" (for the intricate definition consult the literature; for our immediate purposes it is sufficient to know that along geodesics $\kappa_g = 0$) and $K = \frac{1}{R_1 R_2}$ refers to the "Gaussian curvature." On a sphere it is everywhere the case that $K = 1/R^2$, so

$$\iint_D K\, dS = \Omega \quad : \quad \textit{spherical angle subtended by } D$$

For spherical triangles (bounded by geodesics, with interior angles α, β and γ) we recover Heron's formula

$$0 + \Omega = 2\pi - ([\pi - \alpha] + [\pi - \beta] + [\pi - \gamma]) = (\alpha + \beta + \gamma - \pi)$$

while for plane triangles $0 + 0 = (\alpha + \beta + \gamma - \pi)$ gives

$$\alpha + \beta + \gamma = \pi$$

TOPICS YET TO BE TREATED

22. Canonical transformations generated by Stokes' observables.

23. Stokes parameters for quantum oscillators.

24. Stokes parameters for 2-dimensional hydrogen.

25. Wavepackets in elliptical orbit.